ROSICO: Toric Arrangements

I SEMESTRE: varietà toriche affini

↳ Ⓐ Note di Telen: " Introduction to Toric Geometry " (SEZ. 1/2)

II SEMESTRE: Anello di Chow di varietà toriche

↳ Ⓑ Tesi di Christoph Pegel: " Chow Rings of Toric Varieties "

(abbiamo fatto fino a 1.6 della tesi nel I semestre)

PIANO da SEGUIRE

$\boxed{1}$ ⇝ Ⓐ, Sez 3 di Varietà Toriche Proiettive

⇝ Ⓑ, Sez 1.7 / Ⓐ, Sez 4 di Varietà Toriche in senso astratto

$\boxed{2}$ Ⓑ, Cap 2 su Combinatorial Chow Rings

L'idea è di considerare Σ fan e $\Sigma \longrightarrow X_\Sigma$ varietà torica astratta.

Poi $X_\Sigma \longrightarrow A_*(X_\Sigma)$ anello di Chow.

<u>Domanda</u>: si riesce a passare da Σ fan direttamente ad $R(\Sigma)$ (isomorfo all'anello di Chow)? sì.

$$\begin{tikzcd} \underset{\text{fan}}{\Sigma} \arrow[r] \arrow[rrd, bend right] & X_\Sigma \arrow[r] & A_*(X_\Sigma) \arrow[d, equal, "\to \text{Thm. Danilov}"] \\ & & R(\Sigma) \end{tikzcd}$$

$\boxed{3}$ Ⓑ, Cap 3 Teorema di Danilov (no proof nella Tesi).

Per questo si cerca nell'articolo:

Ⓒ Art. Danilov: " The geometry of Toric Varieties "

$\boxed{3'}$ Guardare l'articolo e capire

(D) Leung - Reiner : "The Signature of a Toric Variety"

e un problema aperto è: estendere ?

$\boxed{3''}$ De Concini : come si costruisce per un matroide il Chow ring ?

(Projective) Toric Varieties

▸ **DEF** Affine Toric Variety

Let $T \cong (\mathbb{C}^*)^n$ algebraic torus, $n \geq 1$.

An AFF. TORIC VARIETY is irreducible affine variety V containing a torus $T \subset V$ as a Zariski-dense subset and the inclusion $T \hookrightarrow V$ extends to an action of T on V.

▸ **EXA**

$V \cong \mathbb{C}^n \supseteq (\mathbb{C}^*)^n$ and $(\mathbb{C}^*)^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$.

$$((t_i)_i, (x_i)_i) \longmapsto (t_i x_i)_i$$

▸ **DEF.** Characters of T

$$M = \mathcal{X}(T) = \{\chi : T \to \mathbb{C}^* \text{ algebr. group homom}\} \cong \mathbb{Z}^n$$

$$\left((t_1, \ldots, t_n) \overset{\chi^k}{\longmapsto} t_1^{k_1} t_2^{k_2} \ldots t_n^{k_n}\right) \longleftrightarrow \overset{k}{\overset{\shortparallel}{(k_1, \ldots, k_n)}}$$

⇝ They correspond to Laurent monomials.

Taking a subset $\mathcal{A} = \{m_1, \ldots, m_s\} \subset M$ subset, we associate

$$\begin{aligned} \Phi_{\mathcal{A}} : \quad T &\longrightarrow \mathbb{C}^s \\ t &\longmapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \end{aligned}$$

and by considering its closure:

$Y_{\mathcal{A}} = \overline{\Phi_{\mathcal{A}}(T)} \subset \mathbb{C}^s$ is an affine toric variety with the tori $\Phi_{\mathcal{A}}(T)$ included.

It is still a toric variety because we can see $\Phi_{\mathcal{A}} : T \to (\mathbb{C}^*)^s$ as a morph of tori

▸ **EXA**

- $T \cong \mathbb{C}^*$ and $M = \mathbb{Z} \supset \{2,3\} =: \mathcal{A}$. Then

$$\begin{aligned} \Phi_{\mathcal{A}} : \mathbb{C}^* &\longrightarrow \mathbb{C}^2 \\ t &\longmapsto (t^2, t^3) \end{aligned} \quad \text{and} \quad Y_{\mathcal{A}} = \overline{\Phi_{\mathcal{A}}(T)} = \{x^3 - y^2\} \subseteq \mathbb{C}^2.$$

it's a cusp

- $T \cong (\mathbb{C}^*)^2$ and $\mathcal{A} = \{(1,0),(0,1),(1,1)\}$. Then:

$$\Phi_{\mathcal{A}}: T \longrightarrow \mathbb{C}^3 \qquad (u,t) \longmapsto (u,t,ut)$$

and $Y_{\mathcal{A}} = \{z = xy\}$

DEF. Affine Semigroup.

An affine semigroup S is a semigroup with 1, commutative, finitely generated which embeds in $\mathbb{Z}^n$.

EXA

- Let $\mathcal{A} = \{2,3\} \subseteq \mathbb{Z}$. Then $\mathbb{N}\mathcal{A} = \{0\} \cup \mathbb{Z}_{\geq 2}$ is a semigroup.
- Let $\mathcal{A} = \{(1,0),(0,1),(1,1)\} \subseteq \mathbb{Z}^2$. Then $\mathbb{N}\mathcal{A} = \mathbb{N} \times \mathbb{N}$ is a semigroup.

— o —

DEF. Semigroup Algebra

Semigroup Algebra of S affine semigroup is:

$$\mathbb{C}[S] = \left\{ \sum_{m \in S} c_m \chi^m \;\middle|\; c_m \in \mathbb{C} \text{ fin. many} \neq 0 \right\}.$$

It is an algebra with multiplication $\chi^m \chi^{m'} = \chi^{m+m'}$.

EXA

If $S = \mathbb{N}$ then $\mathbb{C}[S] = \mathbb{C}[x]$.

If $S = \mathbb{Z}$ then $\mathbb{C}[S] = \mathbb{C}[x, x^{-1}]$.

If $S = \mathbb{N}\{2,3\}$ then $\mathbb{C}[S] = \mathbb{C}[x^2, x^3] \subset \mathbb{C}[x]$.

If $S = \mathbb{N} \times \mathbb{N}$ then $\mathbb{C}[S] = \mathbb{C}[x,y]$.

PROPOSITION

If S is a semigroup then $\mathbb{C}[S]$ is a finitely gen. $\mathbb{C}$-algebra and $\operatorname{Spec}(\mathbb{C}[S])$ is an affine toric variety.

· DIM. To show fin. gen. $\mathbb{C}$-alg we have

$$\mathbb{C}[x_1, \dots, x_n]/_{I} \xrightarrow{\sim} \mathbb{C}[S]$$

and so $\mathrm{Spec}(\mathbb{C}[S]) = V(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \ \forall f \in I\}$. ...

► EXA

		Spec ↓
$S = \mathbb{N}$	$\mathbb{C}[S] = \mathbb{C}[x]$	$\mathbb{C}$
$S = \mathbb{Z}$	$\mathbb{C}[S] = \mathbb{C}[x, x^{-1}]$	$\mathbb{C}^*$
$S = \mathbb{N}\{2,3\}$	$\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$	cusp "$x^2 = y^3$"
$S = \mathbb{N} \times \mathbb{N}$	$\mathbb{C}[S] = \mathbb{C}[x, y]$	$\mathbb{C}^2$

► THEOREM

Let V be an affine variety. Then

V is toric $\iff$ $V \cong \mathrm{Spec}(\mathbb{C}[S])$ for S affine semigroup.

— o —

► Rational Cones

∘> σ is a RATIONAL CONE in $\mathbb{R}^n$

∘> $\sigma^\vee$ is the DUAL CONE:

$$\sigma^\vee = \{x \in \mathbb{R}^n \mid \langle u, x \rangle = 0 \ \forall u \in \sigma\}$$

Then $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n$ is an affine semigroup.

OBS: we need a rational cone or else we wouldn't get a fintely generated S_σ.

Therefore we can construct the associated affine toric variety

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

These are special:

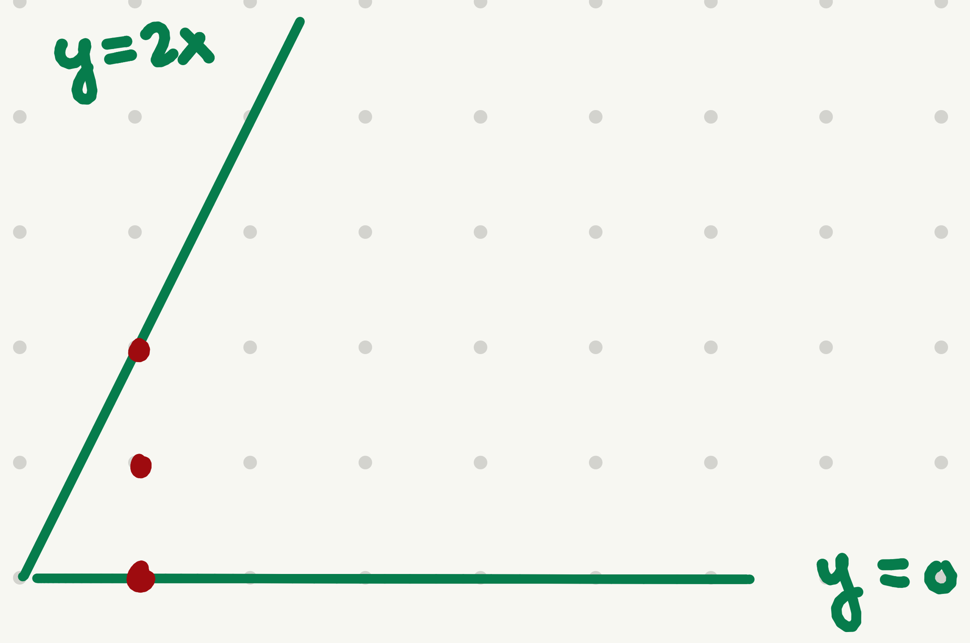
THEOREM

Let V be an affine variety.

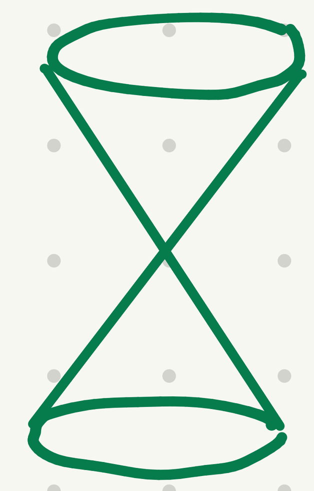
$V \simeq \operatorname{Spec} \mathbb{C}[S_\sigma] \iff V$ is a <u>normal</u> affine toric variety

Avoids some type of singularities like sing of codim 1 (cusp x example)

EXA

Let $\sigma^\vee =$ 

and so $S_\sigma = \mathbb{N}\{(1,0),(1,1),(1,2)\}$.

Then $\mathbb{C}[S_\sigma] = \mathbb{C}[x,y,z] / (y^2 = xz)$ which gives .

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PROJECTIVE TORIC VARIETIES

DEF.

A PROJECTIVE TORIC VARIETY is a projective variety with a Zariski-dense torus such that the action of the torus extends.

EXA

$$\mathbb{P}^n\mathbb{C} = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$$

$$T_{\mathbb{P}^n} = \{(x_0 : \dots : x_n) \mid x_i \neq 0 \ \forall i\} \cong (\mathbb{C}^*)^n \cong (\mathbb{C}^*)^{n+1} / \mathbb{C}^* .$$

Taking $\mathcal{A} = \{m_1, \dots, m_s\} \subset M = \mathfrak{X}(T)$ then we get

$$T \xrightarrow{\Phi_{\mathcal{A}}} (\mathbb{C}^*)^s \xrightarrow{\pi} \mathbb{P}^{s-1}$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Then:

$$X_{\mathcal{A}} := \overline{\operatorname{Im}(\pi \circ \Phi_{\mathcal{A}})}$$ is a PROJECTIVE TORIC VARIETY.

and in particular $X_{\mathcal{A}} \cong \pi(Y_{\mathcal{A}} \setminus \{0\})$.

▸ EXA

Let $\mathcal{A} = \{2,3\} \subset \mathbb{Z}$. Then:

These coords make the composition inj!

$$\begin{array}{ccccc} \mathbb{C}^* & \longrightarrow & (\mathbb{C}^*)^2 & \xrightarrow{\pi} & \mathbb{P}^1 \\ t & \longmapsto & (t^2, t^3) & \longmapsto & \{(1:t)\} \end{array}$$

so the image would be a chart, $X_{\mathcal{A}} = \mathbb{P}^1$.

▸ PROPOSITION

We get that $\dim X_{\mathcal{A}} = \begin{cases} \dim Y_{\mathcal{A}} & \text{otherwise.} \\ \dim Y_{\mathcal{A}} - 1 & \text{if } \mathcal{A} \text{ lie on hyperpl. of } \mathbb{Z}^n. \end{cases}$

• DIM IDEA: we see that when $\mathcal{A}$ lie in hyperplane of $\mathbb{Z}^n$ then we would get $Y_{\mathcal{A}} = \operatorname{Spec}(\mathbb{C}[S]) = \operatorname{Spec}\left(\mathbb{C}[x_i]/I\right)$ with I homogeneous...

▸ EXA

Let $\mathcal{A} = \{(1,0), (1,1), (1,2)\} \subset \{x=1\} \subset \mathbb{Z}^2$. Then:

$$\begin{array}{ccccc} (\mathbb{C}^*)^2 & \xrightarrow{\Phi_{\mathcal{A}}} & (\mathbb{C}^*)^3 & \xrightarrow{\pi} & \mathbb{P}^2 \\ (u,t) & \longmapsto & (u, ut, ut^2) & \longmapsto & \{(1:t:t^2) \mid t \in \mathbb{C}\} \end{array}$$

In fact $Y_{\mathcal{A}} = \operatorname{Spec}\left(\mathbb{C}[x,y,z]/(y^2 - xz)\right)$ so $X_{\mathcal{A}} = \{y^2 = xz\} \subset \mathbb{P}^2$.

▸ AFFINE PIECES

Let $U_i \cong \mathbb{P}^{s-1} \setminus V(x_i) = \{x_i \neq 0\} \cong \mathbb{C}^{s-1}$.

$$(x_1 : x_2 : \_ : x_s) \longmapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_s}{x_i}\right)$$

Let $\mathcal{A} \subset M$ and $X_{\mathcal{A}}$ projective with $|\mathcal{A}| = s$.

What is $X_{\mathcal{A}} \cap U_i = ?$

Take the following:

$$\Phi_{\mathcal{A}_i} : T \xrightarrow{\Phi_{\mathcal{A}}} (\mathbb{C}^*)^s \longrightarrow \mathbb{C}^{s-1}$$

$$t \longmapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t)) \longmapsto (\chi^{m_1 - m_i}(t), \overset{\hat{i}}{\dots}, \chi^{m_s - m_i}(t))$$

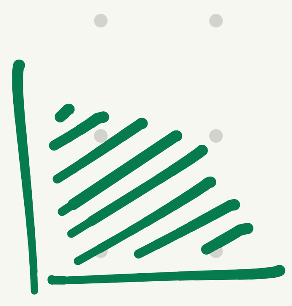
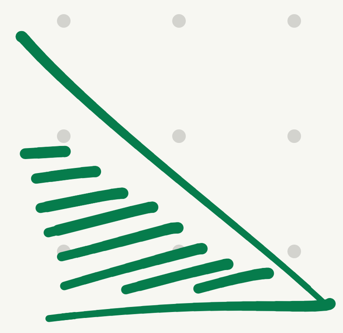
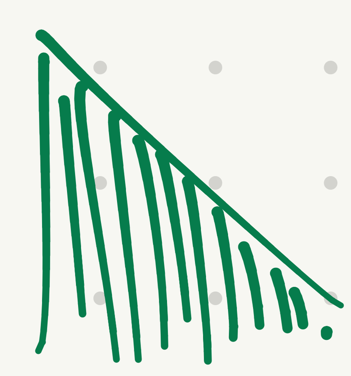
Then we get that
$$X_{\mathcal{A}} \cap U_i \cong Y_{\mathcal{A}_i}.$$

Therefore, we can define
$$S_i = \text{semigroup generated by } \mathcal{A}_i = \mathcal{A} \setminus \{m_i\}$$

and so:
$$X_{\mathcal{A}} \cap U_i \cong \operatorname{Spec} \mathbb{C}[S_i].$$

EXA

Take $\mathcal{A} = \left\{ \begin{matrix} \cdot^{(0,1)} & \\ \cdot^{(0,0)} & \cdot^{(1,0)} \end{matrix} \right\}$ and so we would get:

$S_0 = \{$  $\}$ $S_1 = \{$  $\}$ $S_2 = \{$  $\}$

Dual:   

How are these pieces "glued up together"?

What is $X_{\mathcal{A}} \cap U_i \cap U_j$?
$$X_{\mathcal{A}} = \{ x \in X_{\mathcal{A}} \cap U_i \mid \tfrac{x_j}{x_i} \neq 0 \}$$

and by a construction...

Recall: if $f \in \mathbb{C}[V]$ where V affine variety we get the localization $\mathbb{C}[V]_f = \{ \frac{a}{f^k} \mid a \in \mathbb{C}[V] \}$.

This is the COORDINATE RING of $\{f(x) \neq 0\}$.

EXA

$V \cong \mathbb{C}^2$, $f = x_1$. Then $\{f(x) \neq 0\} = \mathbb{C}^* \times \mathbb{C}$ and so
$$\mathbb{C}[V]_{x_1} = \mathbb{C}[x_1, x_2, x_1^{-1}]$$

With this in mind:

$$X_A \cap U_i \cap U_j = \left\{ x \in X_{\mathcal{A}} \cap U_i \;\middle|\; \frac{x_j}{x_i} \neq 0 \right\}$$

$$\cong \operatorname{Spec}\left(\mathbb{C}[S_i]_{\chi^{m_j - m_i}} \right) \cong \operatorname{Spec}\left(\mathbb{C}[S_j]_{\chi^{m_i - m_j}} \right).$$

EXA

If $X_{\mathcal{A}} = \mathbb{P}^n$ and $\mathcal{A} = \{e_1, -, e_{n+1}\}$ then:

$$X_{\mathcal{A}} \cap U_i = \operatorname{Spec}\left(\mathbb{C}\left[\frac{x_1}{x_i}, -, \frac{x_{n+1}}{x_i} \right] \right)$$

and so:

$$X_{\mathcal{A}} \cap U_i \cap U_j = \operatorname{Spec}\left(\mathbb{C}\left[\frac{x_1}{x_i}, -, \frac{x_{n+1}}{x_i}, \frac{x_i}{x_j} \right] \right) \cong \operatorname{Spec}\left(\mathbb{C}\left[\frac{x_1}{x_j}, -, \frac{x_n}{x_j}, \frac{x_j}{x_i} \right] \right)$$

The inclusion of rings $\mathbb{C}[S_i] \hookrightarrow \mathbb{C}[S_i]_{\chi^{m_j - m_i}}$ will determine the inclusion $X_{\mathcal{A}} \cap U_i \cap U_j \hookrightarrow X_{\mathcal{A}} \cap U_i$.

DEF. Varietà Torica

Sia X varietà algebrica e $T_n \underset{op}{\subseteq} X$ con T_N toro aperto denso con $T_N \simeq (\mathbb{C}^*)^n$ dove $n = \dim X$ e:

$$\begin{tikzcd} T_n \times T_n \arrow[r, "m"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & T_N \arrow[d, hook] \\ T_N \times X \arrow[r] & X \end{tikzcd}$$

Q: come costruisco varietà alg né affini né proiettive?

Sia I insieme finito, $(V_\alpha)_{\alpha \in I}$ varietà algebriche affini:

- consideriamo $V_{\beta\alpha} \underset{op}{\subseteq} V_\alpha$ sottov. alg affine aperta $\forall \alpha, \beta \in I$
- mappe $g_{\beta\alpha} : V_{\beta\alpha} \to V_{\alpha\beta}$ $\forall \alpha, \beta \in I$ tali che:

  i) $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$

  ii) $\forall \alpha, \beta, \gamma \in I$ $\quad g_{\beta\alpha}(V_{\gamma\alpha} \cap V_{\beta\alpha}) = V_{\gamma\beta} \cap V_{\alpha\beta} \subset V_\beta$

  e che $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$.

Questo è rilevante perché si può ora considerare

$$V = \coprod_I V_\alpha /_\sim \quad \text{dove la rel di equivalenza è data da}$$

$\alpha \sim \beta$ se $\exists\, \alpha, \beta \in I$ t.c. $a \in V_{\beta\alpha}$, $b \in V_{\alpha\beta}$ e $g_{\beta\alpha}(a) = b$.

DEF. Varietà Algebrica

Una VARIETÀ ALGEBRICA è uno spazio topologico costruito come sopra.

ESEMPIO

Consideriamo $V_1 = \mathbb{C} = V_2$ e $V_{12} = \mathbb{C}^* \subset V_1$, $V_{21} = \mathbb{C}^* \subseteq V_2$.

(i) Se $\quad g : \underset{V_{21}}{\overset{\mathbb{C}^*}{\shortparallel}} \xrightarrow[z \to z]{} \underset{V_{21}}{\overset{\mathbb{C}^*}{\shortparallel}}$ e si ottiene come spazio V :

Patologico: non è separato.

(ii) Se $\tilde{g}: \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto 1/z$ allora si trova $\mathbb{P}^1_{\mathbb{C}}$.

Come costruiamo varietà toriche astratte?

RECALL: σ cono poliedr razionale

$\downarrow$

$\sigma^\vee$ duale

$\downarrow$

$S_\sigma = \sigma^\vee \cap M$ semigruppo

$\downarrow$

$\mathbb{C} S_\sigma$ dominio, fin gen$/\mathbb{C}$ $\longrightarrow$ $U_\sigma = \mathrm{SpecM}(\mathbb{C}[S_\sigma])$

Lavoreremo con i FAN.

Sia N reticolo, $N \cong \mathbb{Z}^n$ ed $M = N^\vee$. Indichiamo poi:

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \qquad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}.$$

DEF. Un cono poliedrale è FORTEMENTE CONVESSO se 0 è faccia, ovvero se non contiene spazi lineari.

DEF. Fan

Una FAN Σ (razionale fort. convessa) in $N_{\mathbb{R}}$ è una collezione finita di coni σ in $N_{\mathbb{R}}$ tale per cui:

i) i coni σ sono poliedrali razionali e fortemente convessi.

ii) se $\sigma \in \Sigma$ e $\tau \prec \sigma$ allora $\tau \in \Sigma$.

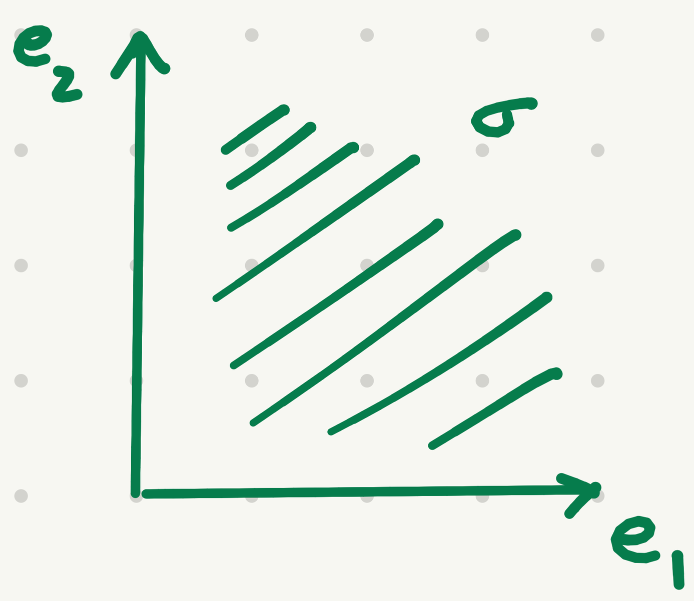
iii) se $\sigma_1, \sigma_2 \in \Sigma$ allora $\sigma_1 \cap \sigma_2 \prec \sigma_1$ e $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

LEMMA

Sia σ cono poliedrale razionale fortemente convesso e $\tau = H_m \cap \sigma$ per $H_m = \{ p \mid \langle m, p \rangle = 0 \}$ che sia una faccia.

Allora $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$.

ESEMPIO



$$\mathbb{C}S_\sigma = \mathbb{C}[x,y]$$

- Sia $\sigma_1 = \operatorname{cone}(e_1) = H_{e_2^*} \cap \sigma$.

Allora dal Lemma:

secondo l'iso

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[x,y,y^{-1}]$$

Infatti ragionando a mano:

$\sigma_1^\vee = \operatorname{cone}(e_1)^\vee = \langle e_1^*, \pm e_2^* \rangle$ e quindi secondo la costruzione $\mathbb{C}[S_{\sigma_1}] \cong \mathbb{C}[\chi^{e_1^*}, \chi^{\pm e_2^*}]$.

- Sia $\sigma_2 = \operatorname{cone}(0)$. Allora $\sigma_2 = H_{e_1^* + e_2^*} \cap \sigma$ e per il Lemma $\mathbb{C}[S_{\sigma_2}] \cong \mathbb{C}[x,y]_{xy} = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Infatti avevamo $\sigma_2^\vee = \mathbb{C} = \operatorname{cone}(\pm e_1^*, \pm e_2^*)$, quindi $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$.

Ricordiamo il seguente fatto.

X affine, $K[X]$ anello delle coordinate (ex: $U_\sigma : K[U_\sigma] = \mathbb{C}S_\sigma$).

$$K[X]_f \longleftrightarrow X \setminus \{f = 0\} = D(f) \underset{op}{\subseteq} X$$

(anello coordinate) (varietà affine associata)

$\Rightarrow$ Localizzare in coordinate corrisponde a prendere un aperto di X.

Se σ è cono e $\tau = H_m \cap \sigma$, allora:

$$U_\tau = \operatorname{SpecM}(\mathbb{C}S_\tau) = \operatorname{SpecM}(\mathbb{C}[S_\sigma]_{\chi^m}) \underset{op}{\subseteq} U_\sigma$$

$\mapsto$ scelta una faccia di σ, ci associo un aperto di σ.

DEF. Varietà X_Σ

Dato Σ un fan, la collezione $(U_\sigma)_{\sigma \in \Sigma}$ con gli aperti

$$U_{\sigma_1,\sigma_2} := U_{\sigma_1 \cap \sigma_2} \underset{op}{\subseteq} U_{\sigma_1}, U_{\sigma_2}$$

e le mappe $g_{\sigma_1,\sigma_2} \equiv id$.

Le condizioni (1), (2) sono verificate quindi chiamo X_Σ la varietà ottenuta incollando.

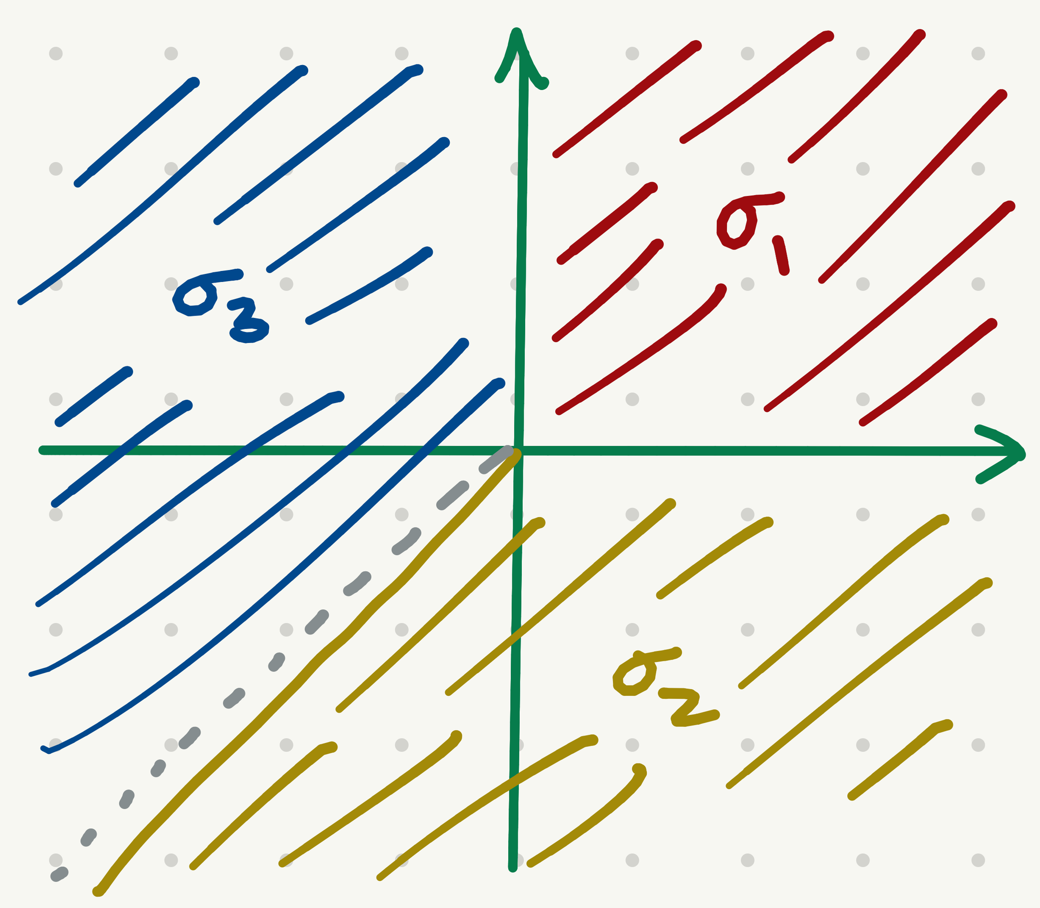
Notiamo che $\{0\}$ è una faccia: vediamo che $U_{\{0\}}$ è il TORO in X_Σ perchè $\{0\}$ è faccia di ogni cono di Σ, quindi $U_{\{0\}} \underset{op}{\subseteq} U_\sigma$ e $U_{\{0\}} \underset{op}{\subseteq} X_\Sigma$.

Siccome le mappe di transizione sono id, l'azione si traspone su X_Σ.

TEOREMA

X_Σ è una varietà torica con il toro denso $U_{\{0\}}$.

ESEMPIO



Siano $\sigma_1 = cone(e_1, e_2)$

$\sigma_2 = cone(e_1, -e_1-e_2)$

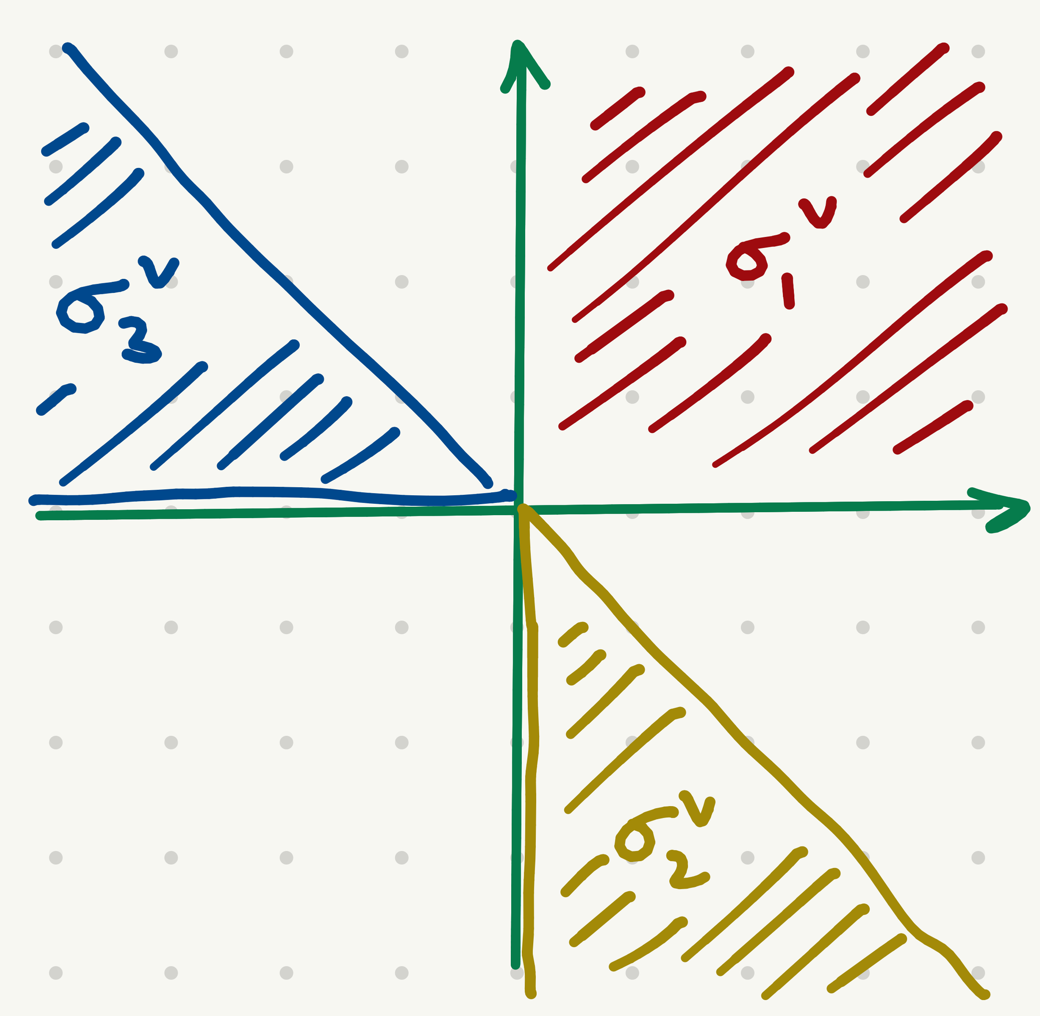
$\sigma_3 = cone(e_2, -e_1-e_2)$.

Calcoliamo i duali.

$\sigma_1^\vee = cone(e_1^*, e_2^*)$

$\sigma_2^\vee = cone(e_1^*-e_2^*, -e_2^*)$

$\sigma_3^\vee = cone(e_2^*-e_1^*, -e_1^*)$



Calcoliamo allora:

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}] \simeq \mathbb{C}[x_1, y_1] \qquad U_{\sigma_1} = \mathbb{C}$$

$$\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[\chi^{e_1^*-e_2^*}, \chi^{-e_2^*}] \simeq \mathbb{C}[x_2, y_2] \qquad U_{\sigma_2} = \mathbb{C}$$

$$\mathbb{C}[S_{\sigma_3}] = \mathbb{C}[\chi^{e_2^*-e_1^*}, \chi^{-e_1^*}] \simeq \mathbb{C}[x_3, y_3] \qquad U_{\sigma_3} = \mathbb{C}$$

con coordinate diverse.

Devo ora incollare $U_{\sigma_1}, U_{\sigma_2}$ lungo $U_{\mathrm{cone}(e_1)}$.

$$\mathrm{cone}(e_1)^\vee = \mathrm{cone}(e_1^*, \pm e_2^*) \implies \mathbb{C}S_{\mathrm{cone}(\sigma_1)} \cong \mathbb{C}[x, y^{\pm 1}]$$

Quindi:

$$\mathbb{C}S_{\mathrm{cone}(\sigma_1)} = \mathbb{C}[\chi^{e_1^*}, \chi^{-e_2^*}, \chi^{e_2^*}] = \begin{cases} = \mathbb{C}[x_1, y_1]_{y_1} \overset{\text{corrisp.}}{\longleftrightarrow} D(y_1) \subseteq U_{\sigma_1} \\ = \mathbb{C}[x_2, y_2]_{y_2} \longleftrightarrow D(y_2) \subseteq U_{\sigma_2} \end{cases}$$

Quindi sto "identificando" $U_{\sigma_1} \cap \{y_1 \neq 0\}$ e $U_{\sigma_2} \cap \{y_2 \neq 0\}$.

Facendo lo stesso iterando ... $\implies X_\Sigma = \mathbb{P}^2_{\mathbb{C}}$.

Prendo le coordinate $(x_0 : x_1 : x_2)$ e prendiamo $\begin{cases} x \mapsto x_1/x_0 \\ y \mapsto x_2/x_0 \end{cases} \rightsquigarrow U_{\sigma_1}$

$$\implies U_{\sigma_i} = \{x_{i-1} \neq 0\} \quad \text{e} \quad U_{\sigma_i} \cap U_{\sigma_j} = \{x_{i-1} \neq 0,\ x_{j-1} \neq 0\}.$$

~ • ~

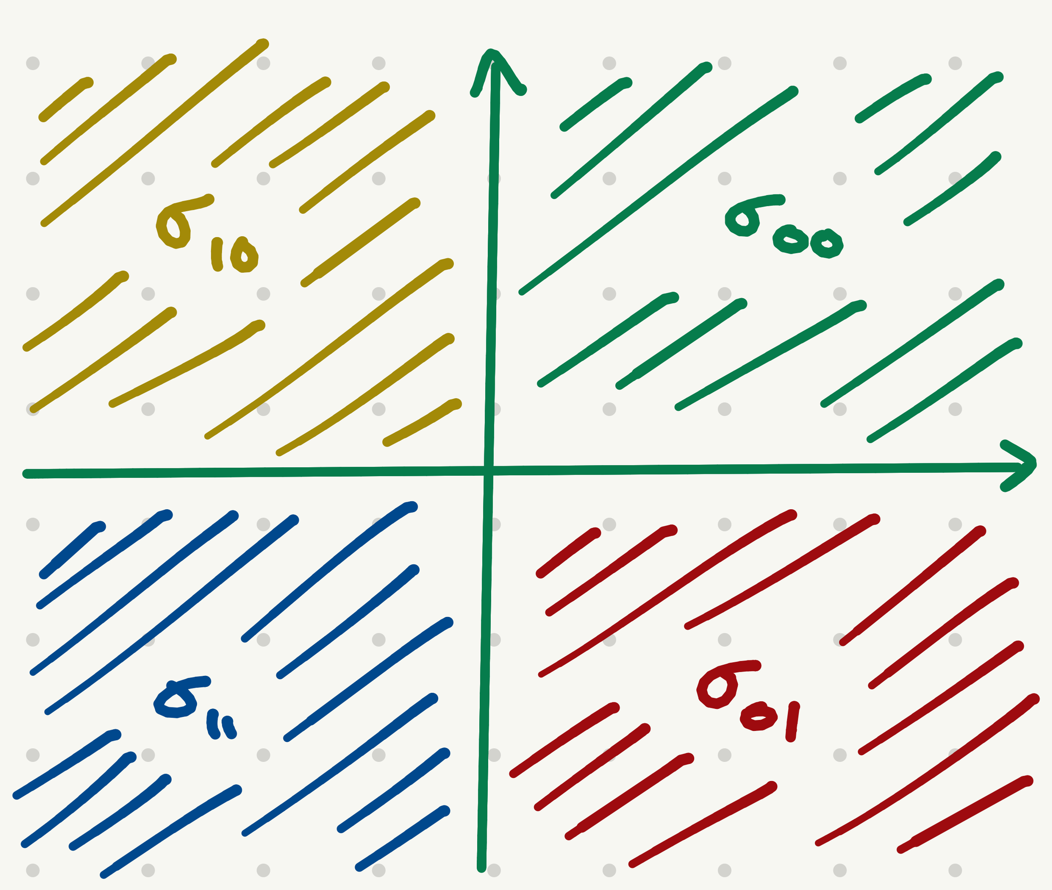
Questo si generalizza con $N = \mathbb{Z}^n$ e considerando

$$e_1, \dots, e_n, \quad e_0 = -e_1 - \dots - e_n$$

prendo Σ fan generato da tutti i sottoinsiemi di $\{e_0, e_1, \dots, e_n\}$.

FATTO: $X_\Sigma = \mathbb{P}^n_{\mathbb{C}}$.

ESEMPIO



$$\mathbb{C}[S_{\sigma_{00}}] = \mathbb{C}[x, y]$$

$$\mathbb{C}[S_{\sigma_{10}}] = \mathbb{C}[x^{-1}, y]$$

$$\mathbb{C}[S_{\sigma_{01}}] = \mathbb{C}[x, y^{-1}]$$

$$\mathbb{C}[S_{\sigma_{11}}] = \mathbb{C}[x^{-1}, y^{-1}]$$

Si vede che $X_\Sigma \simeq \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ e tramite questo isom abbiamo $U_{\sigma_{ij}} = U_i \times U_j$.

DEF. Prodotto di Fan

Siano Σ_1, Σ_2 fan. Allora

$$\Sigma_1 \times \Sigma_2 = \{\sigma \times \tau \mid \sigma \in \Sigma_1, \tau \in \Sigma_2\}$$

è un fan. Vale che $X_{\Sigma_1 \times \Sigma_2} \simeq X_{\Sigma_1} \times X_{\Sigma_2}$.

ESEMPIO

Se nel fan precedente togliamo $\sigma_{0,0}$, stiamo togliendo la carta data da $U_0 \times U_0$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Quindi:

$$X_{\Sigma'} \simeq \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C} \setminus ((0:1),(0:1)).$$

Chow Rings

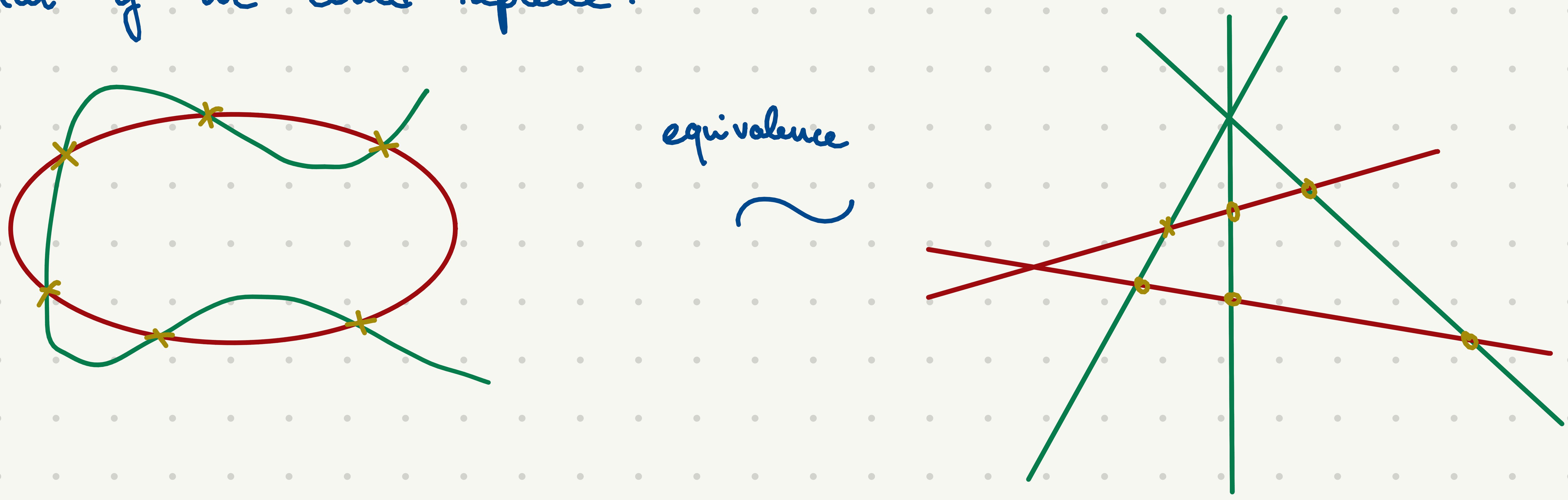
▸ **INTERSECTION THEORY**: study on how varieties intersect.

Bezout's Theorem

Two smooth complex$_1$ curves in $\mathbb{P}^2_2$, which meet transverseley$_3$, of degrees n and m, intersect at $n \cdot m$ points.

We want to describe smoothness without a geometric meaning: homology.

What if we could replace:



▸ "DEF."

Let X be irreducible n-variety.

$$Z_k = \mathbb{Z}\{ Y \subset X \quad k\text{-dimensional} \}$$

$$A_k := Z_k(X) / \mathrm{Rat}(X)$$

Chow group of rational equivalence.

Generalized Bezout's Thm

Let X be a smooth variety, $A, B \subset X$ subvarieties.

To each irreducible component C_i of $A \cap B$ there is a number m_{C_i} such that $[A] \cdot [B] = \sum m_{C_i} [C_i]$.

The Chow group becomes a ring.

OBJ: if X is a TORIC VARIETY, $A(X)$ has a nice combinatorial description: $A^\cdot(X) = H^\cdot(X) =$ combinatorial Chow ring of the fan.

- **From Cones to Curves**

N lattice, σ cone in $N_{\mathbb{R}}$ $\rightsquigarrow \sigma^\vee \rightsquigarrow \operatorname{Spec}(\mathbb{C}[G_\sigma]) = X_\sigma$

- **From Fans to Toric Varieties**

Σ fan $\rightsquigarrow X_\Sigma$ toric variety by gluing $X_{\sigma_1}, X_{\sigma_2}$ along $X_{\sigma_1 \cap \sigma_2}$.

- **From Polytopes to Toric Varieties**

P lattice polytope $\rightsquigarrow \Sigma_P$ normal fan $\rightsquigarrow X_P := X_{\Sigma_P}$

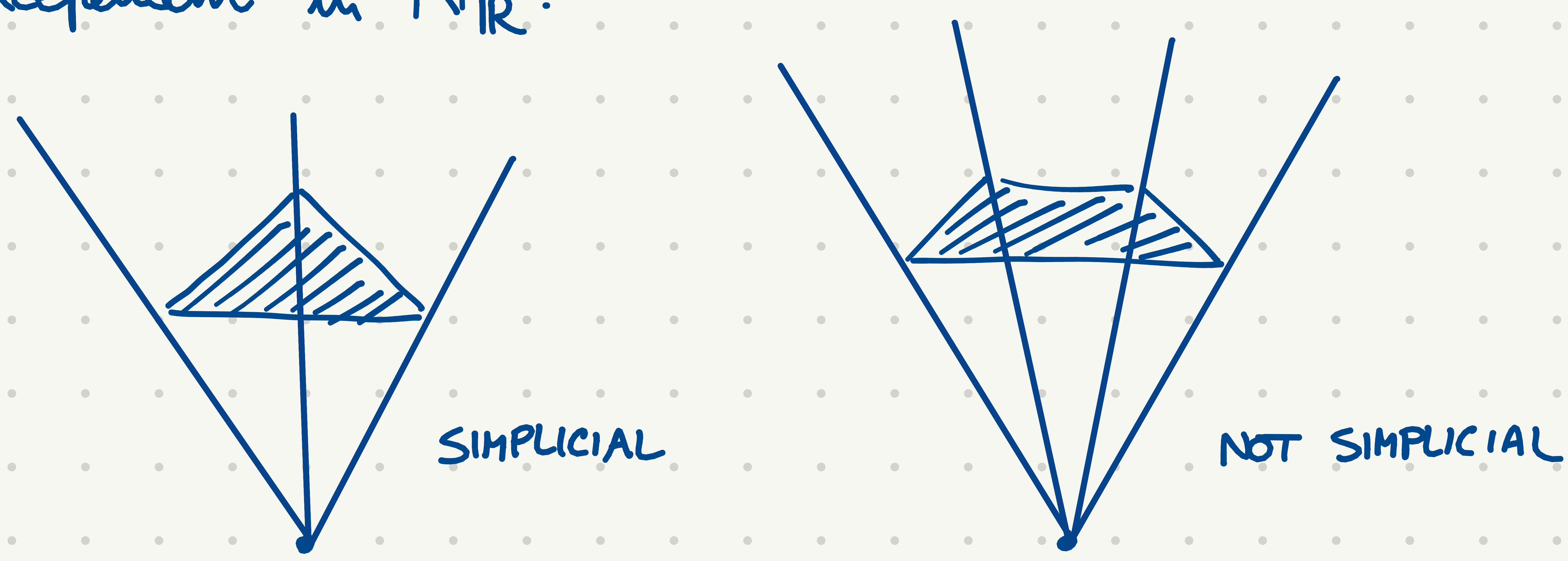
~ · ~

Combinatorial Chow Group

- **DEF.** Simplicial Cone

A cone in $N_{\mathbb{R}}$ is called SIMPLICIAL if the minimal ray generators are linearly independent in $N_{\mathbb{R}}$.

For example:



DEF. Unimodular Cone

A cone in $N_{\mathbb{R}}$ is called UNIMODULAR if the minimal ray generators form a part of the $\mathbb{Z}$-basis of N lattice.

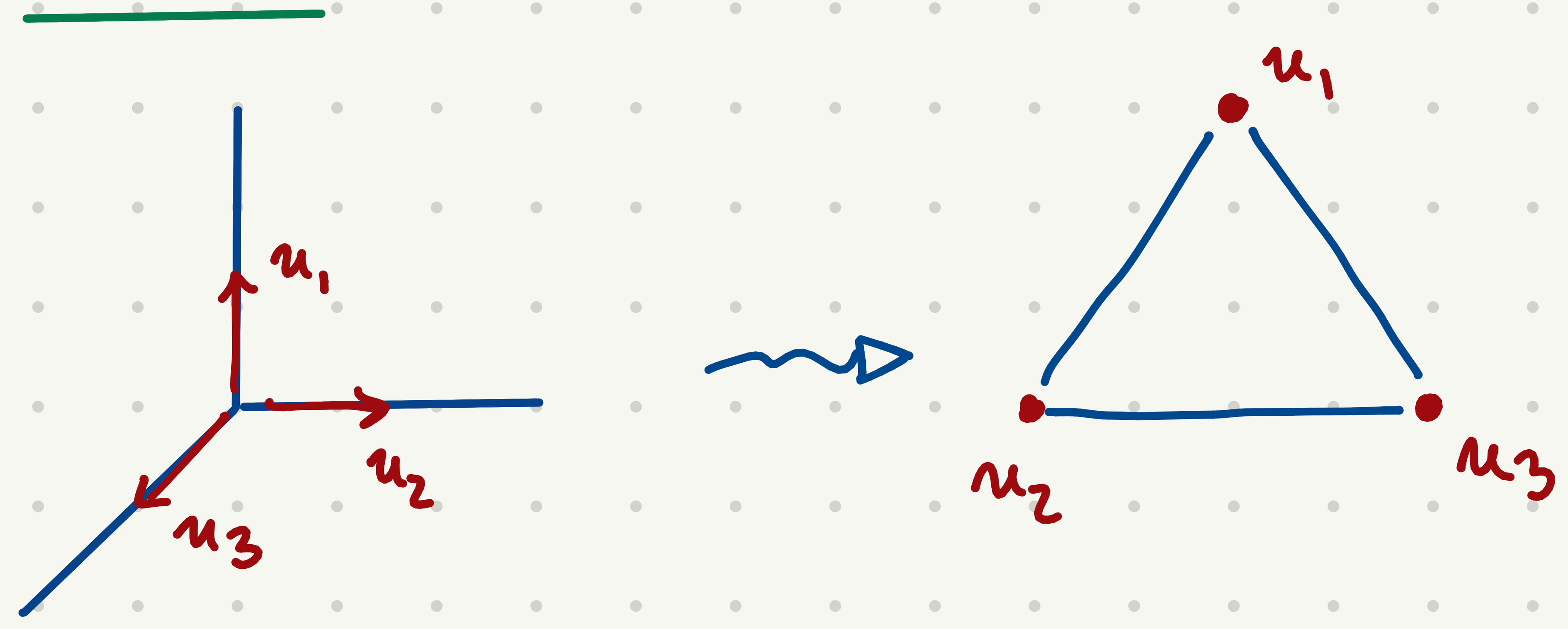
OBS: if $\sigma = \mathbb{R}_{\geq 0}\{u_1, \dots, u_s\}$ in $N_{\mathbb{R}}$ then

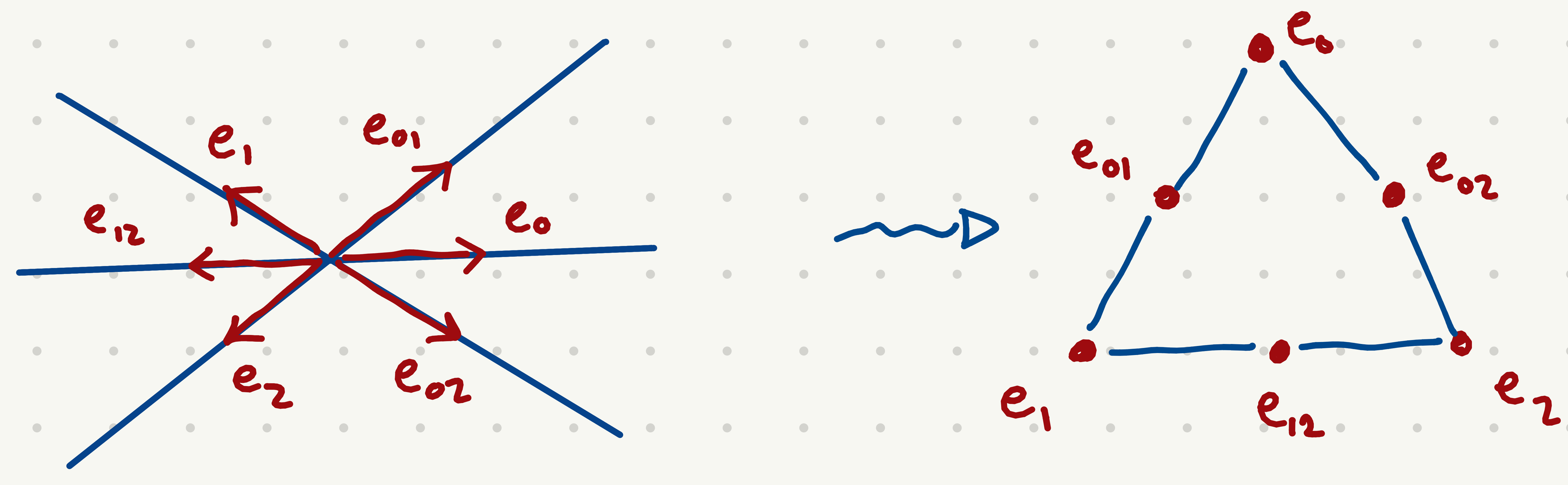
$$\{\text{faces of } \sigma\} \xleftrightarrow{1:1} \{\text{subsets of } \{u_1, \dots, u_s\}\}$$

DEF. Δ_Σ

Let Σ be a simplicial fan with rays $\{\rho_1, \dots, \rho_k\}$. The simplicial complex Δ_Σ on $\{u_1, \dots, u_k\}$ where $\{\text{faces of } \Delta_\Sigma\} \leftrightarrow \{\text{faces of cones in } \Sigma\}$.

EXAMPLE





RECALL Stanley - Reisner

$$\left\{\begin{matrix}\text{simplicial} \\ \text{complexes on } [n]\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{square-free monomials} \\ \text{in } \mathbb{C}[x_1, \dots, x_n]\end{matrix}\right\}$$

$$\Delta \longmapsto I(\Delta) = \langle x_{i_1} \dots x_{i_k} \mid \{i_1, \dots, i_k\} \notin \Delta \rangle$$

STANLEY - REISNER IDEAL

DEF. Combinatorial Chow Group

Let Σ be a simplicial fan in $N_{\mathbb{R}}$.

We denote $\Sigma^{(1)}$ = set of "rays" of Σ. Let

$u_\rho :=$ minimal ray generator of $\rho \in \Sigma^{(1)}$.

The COMBINATORIAL CHOW RING of Σ is

$$R(\Sigma) := \mathbb{Z}[x_\rho : \rho \in \Sigma^{(1)}] \Big/ \left(I(\Sigma) + J(\Sigma) \right)$$

where
$$\begin{cases} I(\Sigma) := I(\Delta_\Sigma) \\ J(\Sigma) := \left\langle \sum_{\rho \in \Sigma^{(1)}} \langle m, u_\rho \rangle x_\rho \ , \ m \in M \right\rangle \end{cases}$$

OBS: one can just choose a basis of M.

We get the following result.

THEOREM

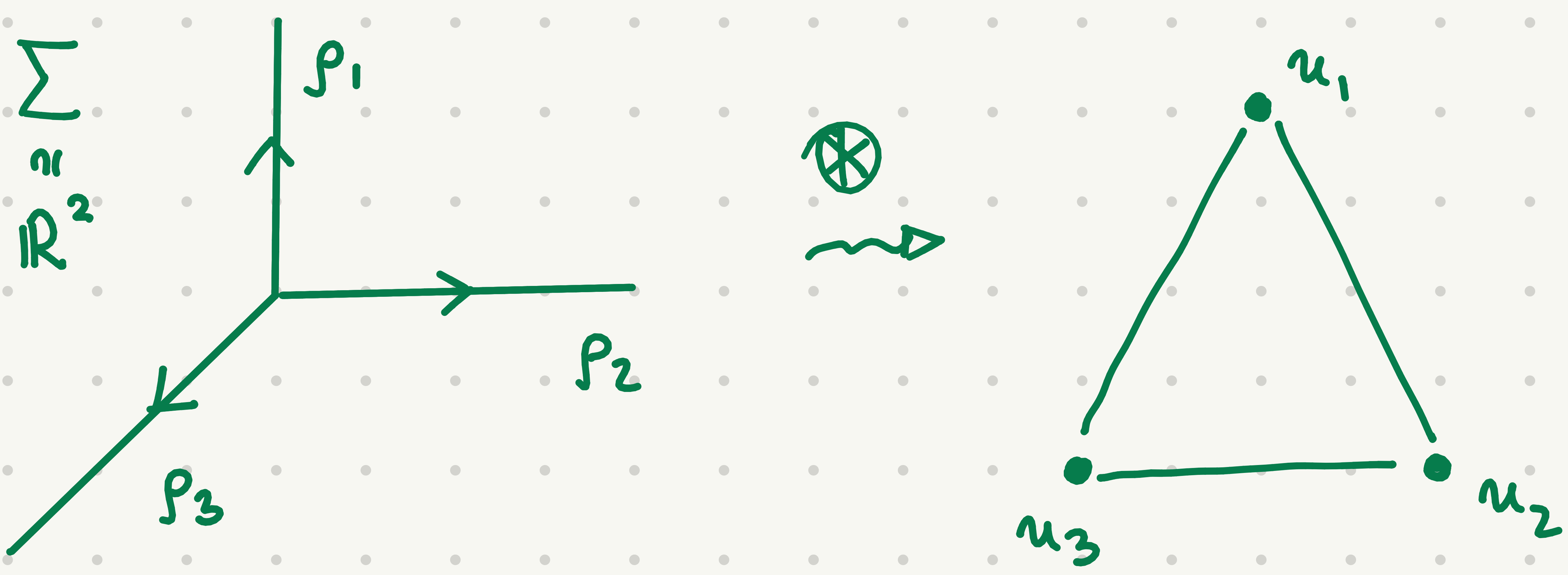
Let Σ be a complete unimodular fan (i.e. X_Σ smooth toric variety), then:

$$R(\Sigma) = A^{\cdot}(X) = H^{\cdot}(X; \mathbb{Q}).$$

Therefore: Betti numbers $\rightsquigarrow$ combinatorics of Σ

Ring structure $\rightsquigarrow$ geometry of Σ.

EXAMPLE



Then $R(\Sigma) = \mathbb{Z}[x_1, x_2, x_3] \Big/ \left(\langle x_1 x_2 x_3 \rangle + J(\Sigma) \right)$.

But $M \cong \mathbb{Z}^2$ is the dual of $N \cong \mathbb{Z}^2$. Hence $M = \langle e_1^*, e_2^* \rangle$.

Therefore we can compute:

$$J(\Sigma) = \Big\langle \langle e_1^*, u_1 \rangle x_1 + \langle e_1^*, u_2 \rangle x_2 + \langle e_1^*, u_3 \rangle x_3 , \quad \langle e_2^*, u_1 \rangle x_1 + \langle e_2^*, u_2 \rangle x_2 + \langle e_2^*, u_3 \rangle x_3 \Big\rangle$$

$$= \langle 1 \cdot x_1 + 0 \cdot x_2 - 1 \cdot x_3 , \ 0 \cdot x_1 + 1 \cdot x_2 - 1 x_3 \rangle = \langle x_1 - x_3 , x_2 - x_3 \rangle .$$

Hence:

$$R(\Sigma) = \mathbb{Z}[x_1, x_2, x_3] \big/ \langle x_1 x_2 x_3 , x_1 - x_3 , x_2 - x_3 \rangle \cong \mathbb{Z}[x] \big/ (x^3) .$$

THEOREM

Let Σ be a unimodular complete fan in $N_{\mathbb{R}} = \mathbb{Z}^n$.

For $\sigma = \rho_{i_1} + \_ + \rho_{i_k} \in \Sigma$ we write:

$$[\sigma] := [x_{\rho_{i_1}} \_ x_{\rho_{i_k}}] \in R(\Sigma).$$

Then we have the following:

$$R(\Sigma) = R^0(\Sigma) \oplus R^1(\Sigma) \oplus \_ \oplus R^n(\Sigma)$$

where $R^i(\Sigma)$ = "subgroup generated by $[\sigma]$, σ an i-dim cone in Σ".

EXAMPLE

In the last example:

$$R(\Sigma) \text{ "=" } \{0\} \oplus \{x_1\} \oplus \{x_1 x_2\}$$ in some sense.

Chow Ring & Chromatic Functions

▸ **DEF.** Root System (cristallographic + finite)

Let V $\mathbb{R}$-vect. space. $\Phi \subseteq V$ is ROOT SYSTEM if:

1) $\mathrm{span}(\Phi) = V$.
2) $\alpha \in \Phi$: $\quad k\alpha \in \Phi$ for $k \in \mathbb{Z}$ $\iff$ $k = \pm 1$.
3) $\alpha, \beta \in \Phi$ then $\beta - \frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha \in \Phi$.
4) $\frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$.

We say $\Phi^+ \subseteq \Phi$ is a CHOICE of subset such that

- $\alpha \in \Phi^+ \iff -\alpha \notin \Phi^+$.
- $\alpha,\beta \in \Phi^+$, $\alpha+\beta \in \Phi \Rightarrow \alpha+\beta \in \Phi^+$.

We say α is SIMPLE ROOT, $\alpha \in \Delta \subseteq \Phi^+$ if

$$\nexists\ \alpha = \beta + \gamma \text{ for } \beta,\gamma \in \Phi^+.$$

▸ **OBS:** $\alpha \in \Phi \Rightarrow \alpha \in \mathrm{span}_{\mathbb{N}}\Delta \cup \mathrm{span}_{\mathbb{N}}(-\Delta)$.

We can define $C_\Delta := \{ v \in V \mid \langle v,\alpha\rangle > 0 \ \forall \alpha \in \Delta\}$ a cone in V.

$\{ C_\Delta \mid \Delta \text{ choice of simple roots}\}$ is DECOMPOSITION of V.

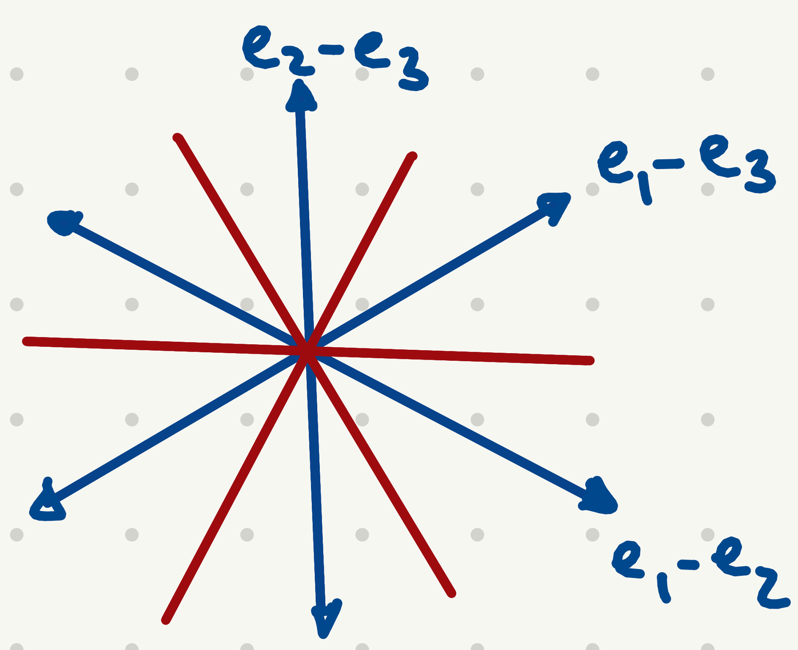
Then we can define the lattice

$$\Lambda = \{ v \in V \mid \langle v,\alpha\rangle \in \mathbb{Z} \ \forall \alpha \in \Phi\}.$$

▸ EXAMPLE

In the case of A_n: $\Phi = \{ e_i - e_j \mid 1 \leq i \neq j \leq n+1\} \subseteq \mathbb{R}^{n+1}$.

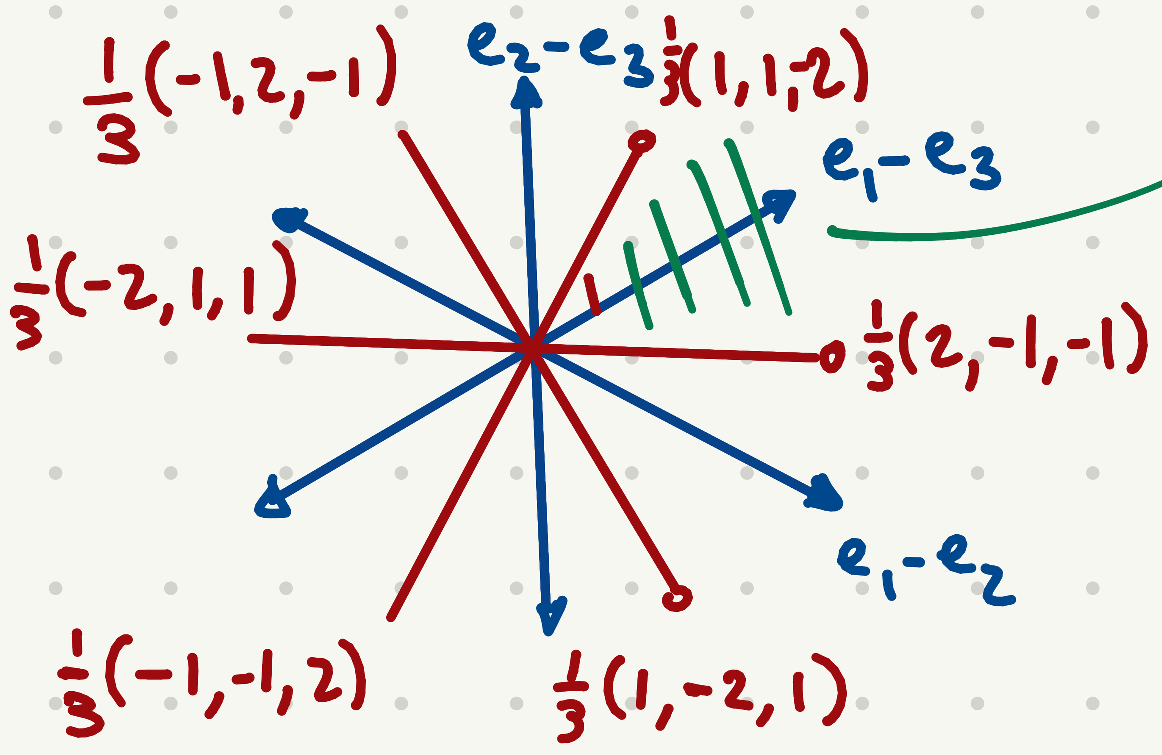
Then for A_2, on the hyperplane $\underline{e}^\perp$:



Therefore, given this decomposition of V into cones, we can construct a TORIC VARIETY. For:

$$A_n \rightsquigarrow \Phi = \{ e_i - e_j \mid i \neq j \} \subseteq \mathbb{R}^{n+1} \longrightarrow X_n$$

▸ EXAMPLE



(Cone corresponding to e_2-e_3 and e_1-e_2.)

Then, we can compute $H^*(X_2)$.

Suppose $x_{12} \longleftrightarrow (1,1,-2)$ and $y_{12} \longleftrightarrow (-1,-1,2)$...

Then:

$$H^*(X_2) = \frac{\mathbb{Q}[x_{12}, x_{13}, x_{23}, y_{12}, y_{13}, y_{23}]}{\langle x_{12}x_{13}, x_{12}x_{23}, x_{12}y_{12}, \text{—} \rangle + \langle x_{13} - y_{13} + y_{23} - x_{23}, \text{—} \rangle}$$

OBS: this is the baricentric subd of last time exa

Fixing $a = x_{12} - y_{12} = x_{13} - y_{13} = x_{23} - y_{23}$ then

$$H^*(X_2) \simeq \frac{\mathbb{Q}[x_{12}, x_{23}, x_{13}, a]}{\langle x_{12}x_{13}, x_{23}x_{13}, x_{12}x_{23}, x_{13}(x_{13}-a), \dots, (x_{13}-a)(x_{12}-a), \dots \rangle}$$

Now, $a^2 = a(x_{12}+x_{23}) = a(x_{12}+x_{13}) = a(x_{13}+x_{23})$ and $x_{12}^2 = a x_{12}$, ...

we get 1 elt in deg 2 and 0 for other degrees...

So with these degrees 1, 4, 1, 0, 0, ...

Looking at the representations in every degree:

$$H^\cdot(X_2) \rightsquigarrow s_3 + q(s_{21} + s_3 + s_3) + q^2 s_3$$

where s_λ is shur funct, for characters.

Now CHROMATIC FUNCTIONS ... $\rightsquigarrow$ (colorings of graph)

For •—•—• we get $q s_{21} + (1+2q+q^2) s_{111}$ and via ω we get

$q s_{21} + (1+2q+q^2) s_3$, which is the result above!

This result extends for all $n \geq 2$: we prove this.

Given X_n is a projective variety...

$A_n = \mathrm{Sym}_{n+1} \curvearrowright \mathbb{P}^n$ by permuting coordinates. For $I \subseteq [0,n]$:

$$\pi_I = \{ x \in \mathbb{P}^n \mid i \in I \Rightarrow x_i = 0 \} \quad \text{so} \quad \sigma \cdot \pi_I = \pi_{\sigma I}$$

$$Z_k = \bigcup_{\#I = n-k} \pi_I \quad \text{union of } \binom{n+1}{k+1} \text{ k-dim planes.}$$

Then by blow-up: $\mathbb{P}^n \leftarrow Y_0 \leftarrow Y_1 \leftarrow \dots \leftarrow Y_{n-2}$ where

Y_0 is blow-up of $\mathbb{P}^n$ along Z_0, Y_i blow-up of Y_{i-1} along $\tilde{Z}_i$.

THM: $X_n \simeq Y_{n-2}$ isomorphism of varieties. (Procesi '96: Toric Variety arising from Weyl chambers...)

...

Combinatorial Chow Rings & Shellability (dalla tesi...)

SCOPO: simplicial fan shellabile $\rightsquigarrow$ base per Chow ring

DEF. Anello di Chow Combinatorio

Sia $\Sigma \subseteq N_{\mathbb{R}}$ fan simpliciale. Siano $\rho_1, \dots, \rho_r$ i raggi di Σ.

Allora $u_1, \dots, u_r$ generatori primitivi di $\rho_1, \dots, \rho_r$.

L' ANELLO di CHOW COMBINATORIO sarà:

$$R(\Sigma) := \mathbb{Z}[X_1, \dots, X_r] \Big/ I_{\Delta(\Sigma)} + J$$

dove $\begin{cases} I_{\Delta(\Sigma)} & \text{ideale di Stanley-Reisner associato a } \Sigma \\ J := \langle \langle m, u_1 \rangle X_1 + \dots + \langle m, u_r \rangle X_r \; : \; m \in M \rangle . \end{cases}$

Sia $R_{\mathbb{Q}}(\Sigma) := R(\Sigma) \otimes \mathbb{Q}$.

Indichiamo x_i immagine nel quoziente di X_i.

PROPOSIZIONE (richiamo)

Ogni monomio $x_1^{a_1} \dots x_r^{a_r}$ è combinazione $\mathbb{Z}$-lineare di squarefree, se Σ è liscio.

$\Rightarrow R(\Sigma)$ è $\mathbb{Z}$-modulo finitamente generato.

— o —

Diamo la definizione base.

DEF. Complesso Simpliciale Shellable

Un complesso simpliciale Δ è SHELLABLE se esiste $F_1, \dots, F_t$ ordinamento delle faccette di Δ (massimali) tale che:

$$\overline{F_1, \dots, F_{k-1}} \cap \overline{F_k} \text{ è puro di dimensione } \dim F_i - 1 \quad \forall k.$$

FATTO

interseco S^n coi raggi ρ e ρ faccio inviluppo per avere P

Se Σ è simpliciale e completo, allora $\Delta(\Sigma) \simeq \partial P$ con P un politopo ed è noto che ∂P shellable, quindi $\Delta(\Sigma)$ shellable.

DEF. Faccia di Restrizione

Sia $F_1, \dots, F_t$ shelling order di Δ. Allora $\forall i$ la FACCIA di RESTRIZIONE è
$$R(F_i) := \{ v \in F_i \mid F_i \setminus \{v\} \in \overline{F_1, \dots, F_{i-1}} \}.$$
Si rivela essere la faccia minimale di F_i che non sta in nessun F_j, $j<i$.

DEF. (equivalente a shellable ~ Björner-Wachs) (Complessi partizionabili)

Sia $F_1, \dots, F_t$ ordine delle faccette di Δ. Allora

$F_1, \dots, F_t$ shelling order $\iff$
- $\Delta = \coprod_{i=1}^{t} [R(F_i), F_i]$ con $[A,B] = \{C \mid A \subseteq C \subseteq B\}$
- $R(F_i) \subseteq F_j \Rightarrow i<j$

TEOREMA

Sia $\Sigma \subseteq N_{\mathbb{R}}$ fan liscio e shellable. Allora i monomi
$$P_{R(\sigma_i)} := \prod_{\rho_i \in R(\sigma_i)} x_i$$
generano l'anello di Chow $\mathcal{R}(\Sigma)$ come gruppo abeliano.

- **DIM.**

Sia $\sigma_1, \dots, \sigma_t$ shelling per Σ. Allora per la caratterizzazione equivalente $\Delta(\Sigma) = \coprod_{i=1}^{t} [R(\sigma_i), \sigma_i]$. Mostriamo che se $\sigma \in [R(\sigma_i), \sigma_i]$ allora il monomio p_σ è combinazione $\mathbb{Z}$-lineare di $P_{R(\sigma_j)}$ con $j \leq i$.

LEMMA Shifting Lemma

Fissiamo $\tau < \sigma \leq \sigma'$. Allora p_σ combinazione $\mathbb{Z}$-lineare di p_{σ_i} con $\dim \sigma_i = \dim \sigma$ e $\tau \leq \sigma_i \not\leq \sigma'$, quando Σ è liscio.

Facciamo un'induzione "al contrario".

- $\underline{i=t}$: sia $\sigma \in (R(\sigma_t), \sigma_t]$. Allora per Shifting Lemma su $\tau = R(\sigma_t)$ e $\sigma' = \sigma_t$. Allora $p_{\sigma_i} = \sum c_i p_{\tilde{\sigma}_i}$ con $R(\sigma_t) < \tilde{\sigma}_i \nleq \sigma_t \; \forall i$.

  Siccome ogni cono contenente $R(\sigma_t)$ è contenuto in σ_t, segue che $p_\sigma = 0$ in $\mathcal{R}(\Sigma)$.

- $\underline{i<t}$: supponiamo $\forall j > i \; \forall \sigma \in (R(\sigma_j), \sigma_j]$ valga che p_σ sia
$$p_\sigma = \sum_{k>j} c_k p_{R(\sigma_k)}, \text{ con } c_k \in \mathbb{Z}.$$
  Fissiamo $\sigma \in (R(\sigma_i), \sigma_i]$. Applico lo shifting a $\tau = R(\sigma_i)$ e $\sigma' = \sigma_i$. Allora:
$$p_\sigma = \sum c_j p_{\tilde{\sigma}_j} \quad \text{dove } R(\sigma_i) < \tilde{\sigma}_j \nleq \sigma_i$$
  e per definizione di $R(\sigma_i)$ segue che $\tilde{\sigma}_i \leq \sigma_k \iff k > i$.

  Ma allora $\tilde{\sigma}_j \in (R(\sigma_j), \sigma_j]$ per cui applico hp induttiva. ■

DOMANDA: i monomi $p_{R(\sigma_i)}$ sono una base?

Per l'anello di Stanley-Reisner: se $\dim \Delta = n-1$ allora
$$\mathrm{Hilb}(\mathbb{Q}[\Delta] : t) = \sum_{i=0}^{n} f_{i-1}(\Delta) \frac{t^i}{(1-t)^i} = \frac{h_0 + h_1 t + \dots + h_n t^n}{(1-t)^n}.$$

TEOREMA

Se Δ è shellable puro allora $\mathbb{Q}[\Delta]$ è Cohen-Macaulay.

In particolare se $\ell_1, \dots, \ell_n \in \mathbb{Q}[\Delta]$ forme lineari, $\dim \Delta = n-1$, tali che $A := \mathbb{Q}[\Delta] / \langle \ell_1, \dots, \ell_n \rangle$ ha dimensione finita su $\mathbb{Q}$, allora
$$\dim_{\mathbb{Q}} (A_i) = h_i.$$

Siamo nelle ipotesi del nostro setup, perchè A corrisponde a $\mathcal{R}(\Sigma)$.

TEOREMA

Sia $F_1, \dots, F_t$ uno shelling di Δ puro, allora
$$|\{R(F_j) \;:\; |R(F_j)| = i\}| = h_i .$$

Riscriviamo il poly di Hilbert sopra come (mandando $t \mapsto \frac{1}{t}$):
$$\sum_{i=0}^{n} f_{i-1}(t-1)^{n-i} = \sum_{i=0}^{n} h_i t^{n-i}.$$

- DIM.

Scriviamo $\Delta = \coprod_{i=1}^{r} [R(F_i), F_i]$. Allora
$$\sum_{i=0}^{n} f_{i-1}(t-1)^{n-i} = \sum_{j=1}^{r} \sum_{R(F_j) \subseteq G \subseteq F_j} (t-1)^{n-|G|}. \qquad \circledast$$

Se $|R(F_j)| = k$ allora:
$$\sum_{R(F_j) \subseteq G \subseteq F_j} (t-1)^{n-|G|} = \sum_{s=k}^{n} \binom{n-k}{s-k}(t-1)^{n-s} = \sum_{s=0}^{n-k} \binom{n-k}{s} (t-1)^{(n-k)-s}$$
$$= ((t-1)+1)^{n-k} = t^{n-k}.$$

Quindi $\circledast = \sum_{k=1}^{r} |\{R(F_j) \mid |R(F_j)| = k\}| \, t^{n-k}$ per cui si eguagliano i coefficienti $|\{R(F_j) \mid |R(F_j)| = k\}| = h_k$. ∎

$\Rightarrow$ I monomi $P_{R(\sigma_i)}$ sono base per $R_{\Delta}(\Sigma)$.

Danilov's Theorem

Chapter 1: Rational Equivalence and Chow Ring

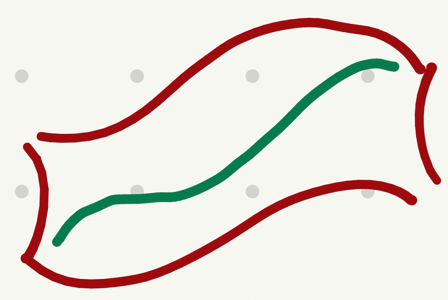
Let X be a smooth n-variety. For $k = 0, \dots, n$ we define

$$Z_k(X) := \mathbb{Z}\{Y \text{ k-subvariety of } X\} \qquad \text{k-CYCLES of } X.$$

We are interested in studying the intersection of these cycles:

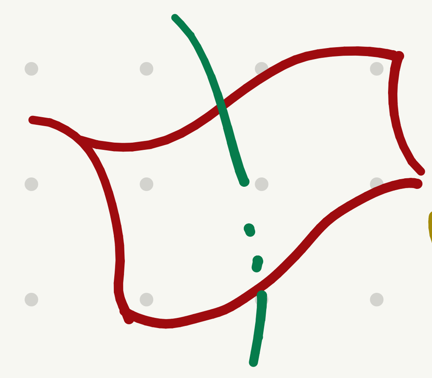
Problem: what's the "codimension" of $Y_1 \cap Y_2$?

NO



We need a concept of "TRASVERSALITY".

To achieve this, we introduce RATIONAL EQUIVALENCE:



YES

let $f : X \to \mathbb{P}^1_{\mathbb{C}}$ be a rational function. We can associate a divisor to f in the following sense:

$$\operatorname{div}(f) := \sum_{\substack{V \subseteq X \\ (n-1)\text{-subvariety}}} \operatorname{ord}_V(f)\,[V] \in Z_{n-1}(X).$$

▸ EXAMPLE $\mathbb{P}^1_{\mathbb{C}}$

Let $X = \mathbb{P}^1_{\mathbb{C}}$ and with coordinates $[z_0 : z_1]$, take the meromorphic function

$$f : \mathbb{C} = U_0 \longrightarrow \mathbb{C} = U_0 \qquad t \longmapsto \frac{t-\alpha}{t-\beta}$$

and extend it to get for $t = \frac{x_1}{x_0}$:

$$\tilde{f} : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}} \qquad [z_0 : z_1] \longmapsto [z_1 - \beta z_0 : z_1 - \alpha z_0]$$

and then $\operatorname{div}(f) = [1 : \alpha] - [1 : \beta]$.

Doing this also in higher dimensions let us define:

$$\operatorname{Rat}_k(X) := \mathbb{Z}\{\operatorname{div}(f) \mid W \subseteq X \ (k+1)\text{-subvar},\ f : W \to \mathbb{P}^1_{\mathbb{C}} \text{ rational}\}$$

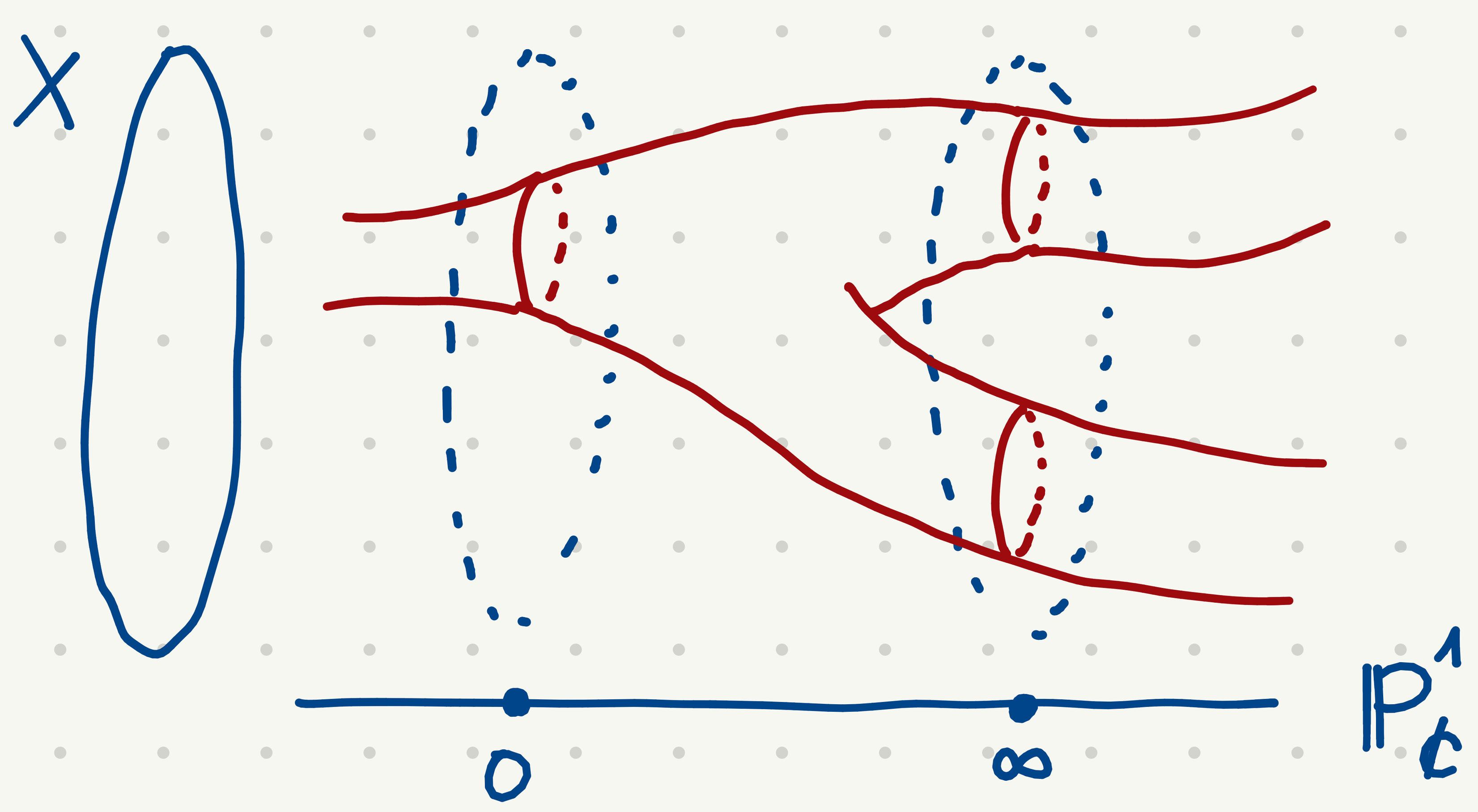
Therefore the CHOW GROUP is defined as

$$A_*(X) = \bigoplus A_k(X) \quad \text{where} \quad A_k(X) = Z_k(X) / \operatorname{Rat}_k(X).$$

▸ EXAMPLE $\mathbb{P}^1_{\mathbb{C}}$

Taking from the previous example, $[P],[Q] \in Z_k(\mathbb{P}^1_{\mathbb{C}})$, if $P=[1:\alpha]$, $Q=[1:\beta]$ then $[P]=[Q]$ in $A_0(\mathbb{P}^1_{\mathbb{C}})$.

For who, like me, is novel to this notion, the book "3264 & all that" by Eisenbud-Harris gives a different visual idea:



Φ subvariety of $X \times \mathbb{P}^1_{\mathbb{C}}$ s.t. $\Phi \not\subseteq X \times \{t\}$ $\forall t \in \mathbb{P}^1_{\mathbb{C}}$.

Then $Rat(X)$ generated by $[\Phi \cap (X \times \{t_0\})] - [\Phi \cap (X \times \{t_\infty\})]$.

This is all done because of the following...

▸ MOVING LEMMA

Let X be a smooth quasi-projective variety. Then

(*) for every $[U],[V] \in A_*(X)$, $\exists\ U' \in [U]$, $V' \in [V]$ such that U' and V' meet trasversely.

(*) the element $[U \cap V]$ does not depend on representatives.

This let us define a PRODUCT on $A_*(X)$. Let

$$A^k(X) := A_{n-k}(X) \quad \text{and} \quad A^*(X) = \bigoplus_{k=0}^{n} A^k(X).$$

Then letting $[U]\cdot[V] := [U \cap V]$ induces by linearity a well defined product.

$\Rightarrow$ $A^*(X)$ is a GRADED RING.

We also now want to take in consideration the following result, which gives a FUNDAMENTAL CLASS for $A^*(X)$...

THEOREM

Let X be irreducible of dimension n. Then
$$A_n(X) \cong \mathbb{Z} \cong \langle [X] \rangle .$$

EXAMPLE Computing $A^*(\mathbb{P}^1_{\mathbb{C}})$

By the Theorem, we know $A^0(\mathbb{P}^1_{\mathbb{C}}) = \langle [\mathbb{P}^1_{\mathbb{C}}] \rangle$.

On the other hand, $Z_0(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{Z}\{P \in \mathbb{P}^1_{\mathbb{C}}\}$ and by the previous example $P \sim Q$ $\forall P, Q \in \mathbb{P}^1_{\mathbb{C}}$. Hence
$$A^1(\mathbb{P}^1_{\mathbb{C}}) = A_0(\mathbb{P}^1_{\mathbb{C}}) = \langle [P] \rangle \simeq \mathbb{Z}.$$

$\boxed{\text{Chapter 2}}$: Generators for $A^*(X_\Sigma)$

Let Σ be a rational fan in $N_{\mathbb{R}}$ lattice (i.e. fan in $N_{\mathbb{Q}}$).

We have constructed X_Σ:
$$\sigma \in \Sigma \text{ in } N_{\mathbb{R}} \Rightarrow \sigma^\vee \text{ in } M_{\mathbb{R}} = N_{\mathbb{R}}^\vee \Rightarrow S_\sigma := \sigma^\vee \cap M .$$

Now we get $U_\sigma := \mathrm{Specm}(\mathbb{C}[S_\sigma])$.

"points of U_σ" $\longleftrightarrow$ "$\gamma : S_\sigma \longrightarrow \mathbb{C}$ semigroup homomorphism"

$p \equiv m_p \longleftrightarrow \quad m \longmapsto \chi^m(p) = p_1^{m_1} \cdot \ldots \cdot p_n^{m_n}$

DEF. Distinguished Point

We define for $\sigma \in \Sigma$ the element $\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$:
$$\gamma_\sigma : S_\sigma \longrightarrow \mathbb{C}$$
$$m \longmapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \cap M \\ 0 & \text{elsewhere} \end{cases}.$$

> **IDEA:** γ_σ is the "0-element" in the affine space $U_\sigma \subseteq \mathbb{C}^{\dim \sigma}$, and elsewhere is non-trivial.

Recall that $T = U_{\{0\}} = \mathrm{Specm}\left(\mathbb{C}[\chi^{\pm e_1^\vee}, \dots, \chi^{\pm e_n^\vee}]\right) \cong (\mathbb{C}^*)^n$.

This gives an action: $\left(T \ni \gamma : (z_1, \dots, z_n) \to z_1 \dots z_n \in \mathbb{C}\right)$

$$U_\sigma \times T \longrightarrow U_\sigma = \mathrm{SpecM}\left(\mathbb{C}[\chi^{m_1}, \dots, \chi^{m_\ell}]\right)$$

$$\left(\gamma, (z_1, \dots, z_n)\right) \longmapsto (z_1, \dots, z_n) \cdot \gamma \quad .$$

We define the ORBIT CLOSURE of σ:

$$\mathcal{O}(\sigma) := T \cdot \gamma_\sigma \subseteq U_\sigma \subseteq X_\Sigma .$$

Now, we can define the fundamental subvarieties we'll use:

$\forall\, \sigma \in \Sigma$ take $\quad F_\sigma := \overline{\mathcal{O}(\sigma)}^{\,zar}$.

These subvarieties have the following properties:

- $\mathrm{codim}(F_\sigma) = \dim(\sigma)$.
- by construction of γ_σ and U_τ invariance of T we get: $\quad F_\sigma \cap U_\tau \neq \emptyset \iff \bar{\tau} \supseteq \sigma$.
- by construction $F_{\{0\}} = T$.

THEOREM Orbit - cone Correspondence

Let X_Σ be a toric variety. Then we have a correspondence:

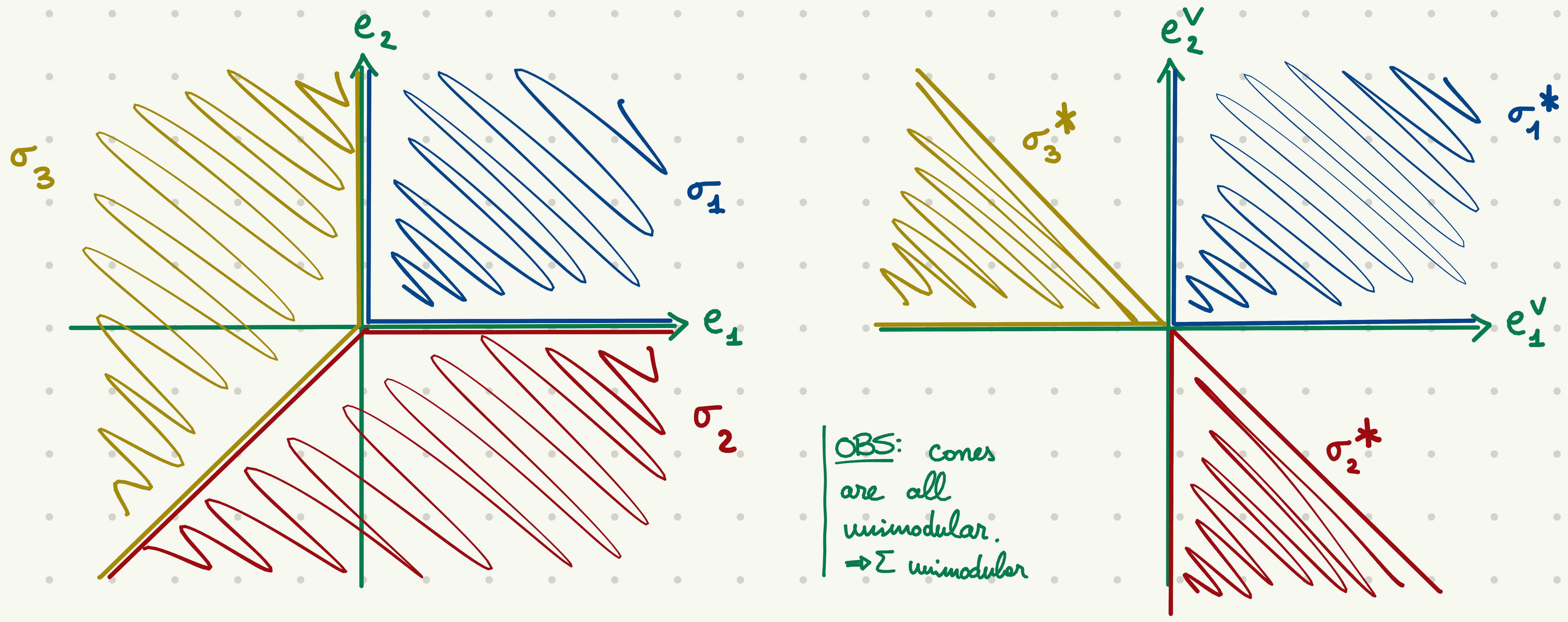
$$\{\text{cones of } \Sigma\} \overset{1:1}{\longleftrightarrow} \{T\text{-orbits in } X_\Sigma\}$$

$$\sigma \longleftrightarrow \mathcal{O}(\sigma) \quad .$$

Now, we will make an example...

▸ EXAMPLE $\mathbb{P}^2_{\mathbb{C}}$

Recall that we've constructed $\mathbb{P}^2_{\mathbb{C}}$ the following way:



Hence, we get the faces:

$$U_{\sigma_1}: \quad \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}] =: \mathbb{C}[x_1, y_1]$$
$$U_{\sigma_2}: \quad \mathbb{C}[\chi^{e_1^*-e_2^*}, \chi^{-e_2^*}] =: \mathbb{C}[x_2, y_2]$$
$$U_{\sigma_3}: \quad \mathbb{C}[\chi^{e_2^*-e_1^*}, \chi^{-e_1^*}] =: \mathbb{C}[x_3, y_3]$$

$x_2 = x_1/y_1 \quad y_2 = 1/y_1$

$x_3 = 1/x_2 \quad y_3 = x_2/y_2$

that glue together to make $\mathbb{P}^2_{\mathbb{C}}$. Fixing coordinates $[z_0 : z_1 : z_2]$:

$$\begin{cases} x_1 = \dfrac{z_1}{z_0} \\ y_1 = \dfrac{z_2}{z_0} \end{cases} \qquad \begin{cases} x_2 = \dfrac{z_1}{z_2} \\ y_2 = \dfrac{z_0}{z_2} \end{cases} \qquad \begin{cases} x_3 = \dfrac{z_2}{z_1} \\ y_3 = \dfrac{z_0}{z_1} \end{cases}$$

$$U_0 = \{z_0 \neq 0\} \qquad U_1 = \{z_2 \neq 0\} \qquad U_2 = \{z_1 \neq 0\}$$

Now, we shall compute torus' orbits.

Let $\sigma = e_1$. Therefore $\gamma_{e_1} \in U_{e_1} = U_{\sigma_1} \cap U_{\sigma_2}$.

$$\gamma_{e_1}: S_{e_1} = \mathbb{N}\{\chi^{e_1^\vee}, \chi^{\pm e_2^\vee}\} \longrightarrow \mathbb{C}$$
$$m \longmapsto \begin{cases} 1 & \text{if } m \in e_1^\vee \cap M = \mathbb{Z} e_2^\vee \\ 0 & \text{elsewhere} \end{cases}$$

In coordinates:

$$\gamma_{e_1} = (0,1) \text{ in } \mathbb{C}^2 \simeq U_0$$

hence $\mathcal{O}(e_1) = T \cdot \gamma_{e_1} = \{(0,t) \mid t \in \mathbb{C}\} \subseteq U_0$.

Therefore $F_{e_1} = \overline{\mathcal{O}(e_1)} = \{z_1 = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$.

By the analogous argument we get

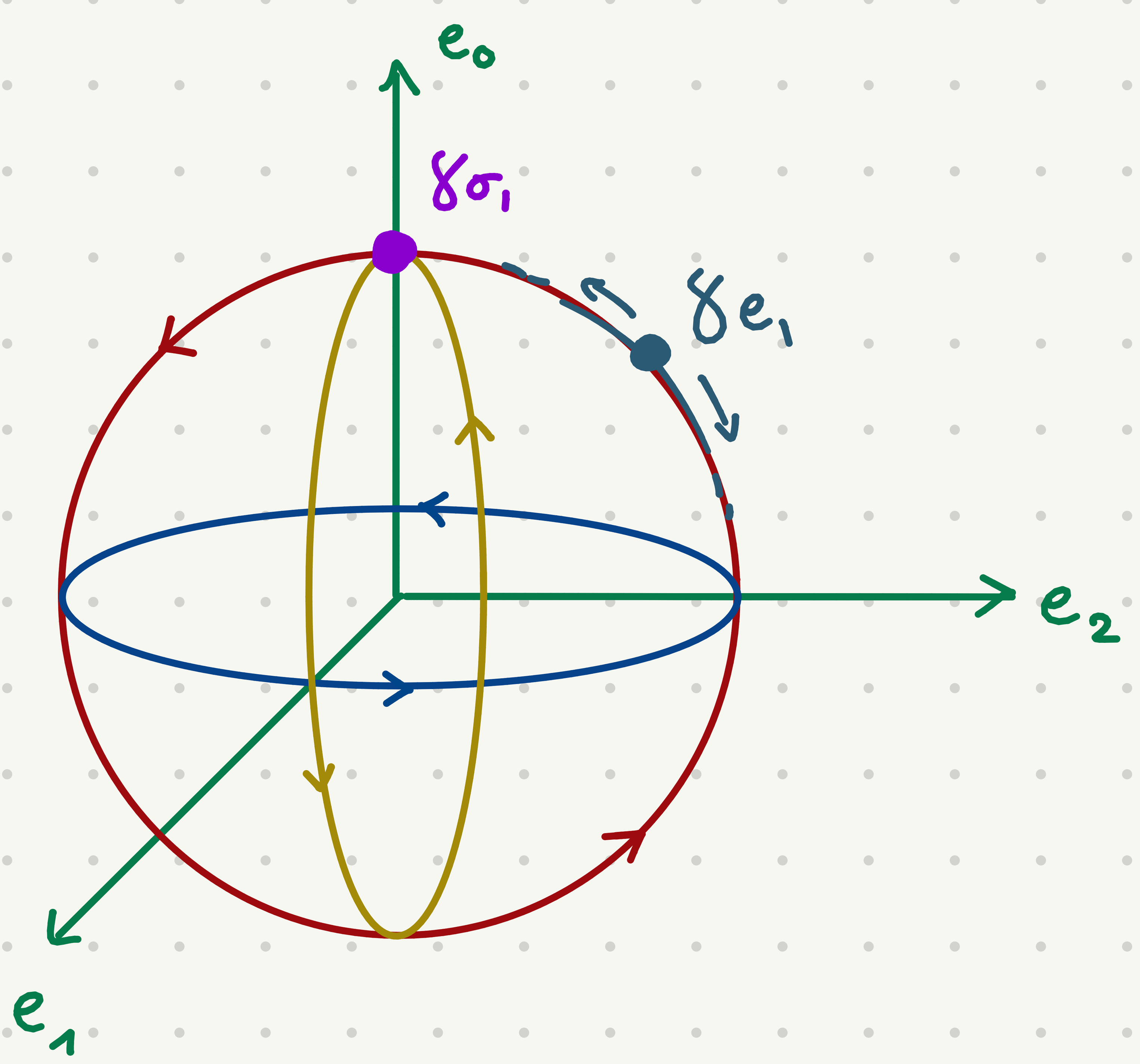
$$F_{e_2} = \overline{\mathcal{O}(e_2)} = \{z_2 = 0\} \quad \text{and} \quad F_{-e_1-e_2} = \{z_0 = 0\}.$$

Considering $\sigma_1 = \mathrm{Cone}\{e_1, e_2\}$, we see that $\sigma_1^\vee = \{0\}$.
Hence in the chart $U_0 \cong U_{\sigma_1}$ we have $\gamma_{\sigma_1} = (0,0) \in U_0$
and so $F_{\sigma_1} = \overline{\mathcal{O}(\sigma_1)} = \{z_1 = z_2 = 0\}$.

Observe that:

$$F_{\sigma_1} = \{z_1 = 0\} \cap \{z_2 = 0\} = F_{e_1} \cap F_{e_2}.$$

Let's try to visualize this. Consider the projection onto the real $\mathbb{P}^2_{\mathbb{R}} \subseteq \mathbb{R}^3$.



Identify the action of the "real torus" as:

$$\begin{aligned} (\mathbb{R}^*)^2 &\longrightarrow \mathbb{R}^3 \\ (\alpha, \beta) &\longrightarrow \big((1,1,1) \to (\alpha, \beta, 1)\big) \end{aligned}$$

Hence, if M is $\mathrm{Span}(e_1^\vee, e_2^\vee)$:

$$\begin{cases} \alpha \cdot e_1 \leftrightarrow (\alpha, 0) \\ \alpha \cdot e_2 \leftrightarrow (0, \alpha) \\ \alpha \cdot e_3 \leftrightarrow (-\alpha, -\alpha) \end{cases}$$

This happens because $\mathbb{P}^2_{\mathbb{R}} \simeq \mathbb{R}^3 / \mathbb{R}^*$

Now, take $U_0 \simeq U_{\sigma_1}$ in $\mathbb{P}^1_{\mathbb{C}}$. This is the north semisphere.
Given $\gamma_{e_1} := (0,1)$ in U_0, in ▇, the action of the torus lets it move on ○ $\mapsto F_{e_1} = \{z_1 = 0\}$.
Given $\gamma_{\sigma_1} := (0,0)$ in U_0, in ▇, it cannot move.
$\mapsto F_{\sigma_1} = \{z_1 = z_2 = 0\}$.

▸ THEOREM

Let X_Σ as above. Then the set $\{[F_\sigma]\}_{\sigma\in\Sigma}$ generate $A_*(X)$.

LEMMA Let $Y \subset X$ be a closed subvariety. The following is exact:

$$\begin{array}{ccccccc} A_*(Y) & \longrightarrow & A_*(X) & \longrightarrow & A_*(X\setminus Y) & \longrightarrow & 0 \\ [V] & \longmapsto & [V \subseteq X] & \longmapsto & [V\cap(X\setminus Y)] & & \end{array}$$

- Proof We get the following steps:
  - $T = F_{\{0\}}$ is an open set in $U_{\{0\}} \subseteq \mathbb{A}^n$. Hence $A_*(T) \cong \langle [T] \rangle$. non-trivial just in A_n
  - Let $Y = X \setminus T$. Then by Lemma we have:

$$A_*(X\setminus T) \longrightarrow A_*(X) \longrightarrow A_*(T) \longrightarrow 0$$

Looking at the degrees:

→ degree n: since T dense in X, $\dim(X\setminus T) < \dim(X)$ hence $A_n(X\setminus T) = 0$. By exactness it follows

$$\begin{array}{ccc} A_n(X) & \longrightarrow & A_n(T) \\ [X] & \longmapsto & [T] \end{array}$$

is an isomorphism.

→ degree $k<n$: by step 1 we know $A_k(T) = 0$. Therefore $A_k(X\setminus T) \twoheadrightarrow A_k(X)$ surjective. By construction, we know that $X\setminus T \subset \bigcup_{\sigma\neq\{0\}} F_\sigma$. Hence, any cycle in $A_k(X)$ actually lives in the cones $\{F_\sigma\}_{\sigma\neq\{0\}}$.

By an induction on $\dim X$, we conclude. ■

$\boxed{\text{Chapter 3}}$: $R(X_\Sigma) \cong A^*(X_\Sigma)$

We take the considerations of the previous chapters to describe explicitly $A^*(X_\Sigma)$. Let Σ be smooth (i.e. UNIMODULAR).

Consider for $\Sigma^{(1)} = \{\sigma_1, -, \sigma_k\}$ and identify

$$\sigma_i \longleftrightarrow \text{PRIMITIVE RAY} \quad u_i \in \sigma_i \cap N.$$

Recall also that $F_\sigma \cap U_\tau \neq \emptyset \iff \tau \supseteq \sigma$.

We have the following observations:

1. if $\{i_1, -, i_h\} \in \Sigma$, so they generate a cone, then:

$$F_{u_{i_1}} \cap \_ \cap F_{u_{i_h}} = F_{<\sigma_{i_1}, -, \sigma_{i_h}>}.$$

This is true by looking at the orbits $\mathcal{O}(u_i)$ in the affine chart $U_{<\sigma_{i_1}, -, \sigma_{i_h}>}$.

2. if $\{i_1, -, i_h\} \notin \Sigma$, then we get:

$$F_{u_{i_1}} \cap - \cap F_{u_{i_h}} \neq \emptyset \iff F_{u_{i_1}} \cap - \cap F_{u_{i_h}} \cap U_\tau \neq \emptyset \text{ for } \tau \in \Sigma$$
$$\iff \sigma_{i_1}, \dots, \sigma_{i_h} \subseteq \tau \quad \text{↯}$$

hence $F_{u_{i_1}} \cap \_ \cap F_{u_{i_h}} = \emptyset$, given Σ <u>unimodular</u>.

3. consider $m \in M$. Then we have the homomorphism

$$\varphi_m : T \longrightarrow \mathbb{C}$$
$$(t_1, -, t_n) \longmapsto t_1^{m_1} \dots t_n^{m_n}$$

that extends to $\overline{\varphi}_m : X_\Sigma \longrightarrow \mathbb{P}^1_{\mathbb{C}}$. This means:

$$\operatorname{div}(\varphi_m) = \sum_{i=1}^{k} m(u_i) [F_{u_i}].$$

All these relations must be contained in $\operatorname{Rat}(X_\Sigma)$.

Considering the ring structure on $A^*(X_\Sigma)$ we get that:

ⓐ $\rightsquigarrow$ $A^*(X_\Sigma)$ is generated by $[F_{e_i}]$ as a ring.

By associating $[F_{e_i}] \longleftrightarrow X_i$ variable, we get a map: $\mathbb{Z}[X_i] \twoheadrightarrow A^*(X_\Sigma)$.

ⓑ $\rightsquigarrow$ If $\langle e_{i_1}, -, e_{i_k} \rangle \notin \Sigma$ then $F_{e_{i_1}} \cap - \cap F_{e_{i_k}} = \emptyset$ means that $X_{i_1} - X_{i_k}$ goes to 0 in the Chow ring.

Such elements identify:

$$\mathbb{Z}\{X_{i_1} - X_{i_k} \mid \{i_1, -, i_k\} \notin \Sigma\} = I(\Delta_\Sigma).$$

ⓒ $\rightsquigarrow$ The divisors of φ^m are in $Rat(X)$, therefore by the correspondence we can quotient by

$$\mathbb{Z}\{\sum_i m(e_i) X_i \mid m \in M\} = J.$$

This is the COMBINATORIAL CHOW RING:

$$R(X_\Sigma) = \mathbb{Z}[X_i] / _{I(\Delta_\Sigma) + J} \twoheadrightarrow A^*(X)$$

is a SURJECTIVE RING HOMOMORPHISM.

▸ EXAMPLE $\mathbb{P}^2_{\mathbb{C}}$

Take the previous example. Here we have:

$$F_{e_1} = \{z_1 = 0\} \qquad F_{e_2} = \{z_2 = 0\} \qquad F_{-e_1-e_2} = \{z_0 = 0\}.$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$X_1 \qquad\qquad X_2 \qquad\qquad X_3$$

and $R(\Sigma) := \mathbb{Z}[X_1, X_2, X_3] / _{\langle X_1 X_2 X_3 \rangle + \langle X_1 - X_3, X_2 - X_3 \rangle}$.

② We know $F_{e_1} \cap F_{e_2} \cap F_{-e_1-e_2} = \{z_1 = z_2 = z_0 = 0\} = \emptyset$.

③ Consider the relation $X_1 - X_3$ generated when $m = e_1^\vee$.

In this case we can look at the chart U_0 where

$U_0 \equiv U_{\sigma_1} \leftrightarrow \mathbb{C}[x_1, y_1]$ and $x_1 = \frac{z_1}{z_0}$, $y_1 = \frac{z_2}{z_0}$.

Then, since $x_1 \sim \chi^{e_1^\vee}$ and $y_1 \sim \chi^{e_2^\vee}$ we have that

$$\begin{aligned} \varphi_{e_1^\vee} : \mathbb{C}^2 \cong U_{\sigma_1} &\longrightarrow \mathbb{C} \\ (x_1, y_1) &\longmapsto x_1^1 \cdot y_1^0 = x_1 \end{aligned}$$

Extending for $\mathbb{P}^2_{\mathbb{C}}$ with $[z_0 : z_1 : z_2] = [1 : x_1 : x_2]$

$\mathbb{P}^1_{\mathbb{C}}$ with $[u_0 : u_1] = [1 : t]$

we find $\overline{\varphi}_{e_1^\vee} : \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ where $0 \equiv [1:0]$, $\infty \equiv [0:1]$

$$[z_0 : z_1 : z_2] \longmapsto [z_0 : z_1]$$

so $\operatorname{div}(\overline{\varphi}_{e_1^\vee}) = \{z_1 = 0\} - \{z_0 = 0\} = F_{e_1} - F_{-e_1 - e_2}$.

At this point, one can prove the following theorem:

THEOREM (Danilov)

Let Σ be a complete smooth fan. The ring homomorphism:

$$R(\Sigma) \xrightarrow{\sim} A^*(X_\Sigma)$$

is an isomorphism and we also have $A^*(X_\Sigma) \simeq H^*(X_\Sigma, \mathbb{Z})$.

- Proof (idea)

The idea is to show that:

→ $R(\Sigma)$ is torsion-free and it is Cohen-Macaulay.

→ we have an isomorphism $A^*(X_\Sigma) \simeq H^*(X_\Sigma, \mathbb{Z})$ which uses Poincaré Duality (for Σ simplicial, construction works on $\mathbb{Q}$) and the case of Σ projective: SHELLABILITY!

Then one concludes by induction.

→ One observes that $\mathrm{rk}(R(\Sigma))$ and $\mathrm{rk}(H^*(X_\Sigma, \mathbb{Z}))$ are equal, both to $a_n = \#\Sigma^{(n)}$.
⇝ HILBERT POLYNOMIAL for $R(\Sigma)$
⇝ EULER CHARACTERISTIC for $H(X_\Sigma, \mathbb{Z})$

This concludes. □

FULTON's ⊛

Take $\sigma_1, -, \sigma_m$ and τ_i = inters. of σ_i with σ_j, $j>i$. Then

(i) $\tau_i \subseteq \sigma_j \iff i \leq j$

(ii) $\Sigma = \coprod [\tau_i, \sigma_i]$

DANILOV

Σ projective fan: g piecewise linear function on Σ.

$x_0 \in N_{\mathbb{R}}$ general position, $\sigma \leq \sigma'$ if $g|_\sigma(x_0) \leq g|_{\sigma'}(x_0)$.